Univerzitet u Zenici
Ekonomski fakultet
Odsjek: Menadžment preduzeća, Računovodstveni i revizijski menadžment
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Sadržaj sveske sa vježbi iz **Matematike**

(I dio sveske - sadrži gradivo od 1 do 8 sedmice)

Literatura za dodatno istraživanje:

- Matematika I za ekonomiste; Zečić, Huskanović, Alajbegović
- Zbirka zadataka iz više Matematike; Uščumlić, Miličić
- Zadaci i riješeni primjeri iz više matematike s primjenom na tehničke nauke; Demidović
- Zbirka zadataka iz Matematike; Stojanović
- Zbirka zadataka iz Matematičke analize; Berman

Dio tablice izvoda

1) $(c)' = 0;$
2) $(u+v-w)' = u' + v' - w';$
3) $(uv)' = u'v + v'u;$
3a) $(cu)' = cu';$
4) $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2};$
4a) $\left(\frac{u}{c}\right)' = \frac{u'}{c};$
4б) $\left(\frac{c}{v}\right)' = -\frac{cv'}{v^2};$
5) $(x^n)' = nx^{n-1};$
6) $(\sin x)' = \cos x;$
7) $(\cos x)' = -\sin x;$
8) $(\operatorname{tg} x)' = \sec^2 x;$
9) $(\operatorname{ctg} x)' = -\operatorname{cosec}^2 x.$

5) $(u^n)' = nu^{n-1} \cdot u';$
6) $(\sin u)' = \cos u \cdot u';$
7) $(\cos u)' = -\sin u \cdot u';$
8) $(\operatorname{tg} u)' = \sec^2 u \cdot u';$
9) $(\operatorname{ctg} u)' = -\operatorname{cosec}^2 u \cdot u'.$

10) $(a^u)' = a^u \ln a \cdot u';$
11) $(\log u)' = \frac{u'}{u} \log e;$
10a) $(e^u)' = e^u u';$
11a) $(\ln u)' = \frac{u'}{u};$
10б) $(a^x)' = a^x \ln a;$
11б) $(\log x)' = \frac{1}{x} \log e;$
10в) $(e^x)' = e^x;$
11в) $(\ln x)' = \frac{1}{x}.$

12) $(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}};$
12a) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}};$
13) $(\arccos u)' = -\frac{u'}{\sqrt{1-u^2}};$
13a) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}};$
14) $(\operatorname{arc\,tg} u)' = \frac{u'}{1+u^2};$
14a) $(\operatorname{arc\,tg} x)' = \frac{1}{1+x^2};$
15) $(\operatorname{arc\,ctg} u)' = -\frac{u'}{1+u^2};$
15a) $(\operatorname{arc\,ctg} x)' = -\frac{1}{1+x^2}.$

Dio tablice integrala

1. $\int u^a\,du = \frac{u^{a+1}}{a+1} + C, \quad a \neq -1.$
2. $\int u^{-1}\,du = \int \frac{du}{u} = \int \frac{u'}{u}\,dx = \ln|u| + C.$
3. $\int a^u\,du = \frac{a^u}{\ln a} + C; \ \int e^u\,du = e^u + C.$
4. $\int \sin u\,du = -\cos u + C.$
5. $\int \cos u\,du = \sin u + C.$
6. $\int \sec^2 u\,du = \operatorname{tg} u + C.$
7. $\int \operatorname{cosec}^2 u\,du = -\operatorname{ctg} u + C.$
8. $\int \frac{du}{u^2+a^2} = \frac{1}{a} \operatorname{arc\,tg} \frac{u}{a} + C.$
9. $\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C.$
10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C.$
11. $\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u + \sqrt{u^2+a}\right| + C.$

Matrice

Neka su m i n pozitivni cijeli brojevi.
$m \times n$ matrica je kolekcija od $m \cdot n$ brojeva uređenih u pravougaoni niz:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ m redova}$$

n kolona

Npr. $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -5 \end{bmatrix}$ je 2×3 matrica, $A = \begin{bmatrix} 1 & \sqrt{2} & 8 & 9 \\ 7 & 2 & -5 & 3 \\ 4 & -6 & 7 & 8 \\ 3 & 7 & 2 & 8 \\ 1 & 2 & -2 & 5 \end{bmatrix}_{5 \times 4}$

Brojeve u matrici zovemo elementi matrice i označavamo sa a_{ij}, gdje su i, j cijeli $1 \le i \le m$ i $1 \le j \le n$. Indeks i zovemo red indeks, a j kolona indeks.

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \dots & a_{ij} & \dots \\ & \vdots & \end{bmatrix} \end{array}$$

Npr. u matrici A

$a_{12} = \sqrt{2}$, $a_{23} = -5$, $a_{43} = 2$, $a_{53} = -2$

$1 \times n$ matricu zovemo n-dimenzionalni red vektor, $A = [a_1 \dots a_n]$
$m \times 1$ matrica je m-dimenzionalni kolona vektor $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Sabiranje matrica: $[a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [s_{ij}]_{m \times n}$,
gdje je $s_{ij} = a_{ij} + b_{ij}$, $\forall i,j$

npr.

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 4 & 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 4 & 1 & 3 & 4 \end{bmatrix}$$

Skalarno množenje matrice brojem:
c je realan broj $\quad c \cdot [a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$
gdje je $b_{ij} = c \cdot a_{ij}$ $\forall i,j$

npr.

$$2 \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

Brojeve ćemo često zvati skalari.

Množenje matrica:
Prvo ćemo vidjeti šta je proizvod red vektora A i kolone vektora B.

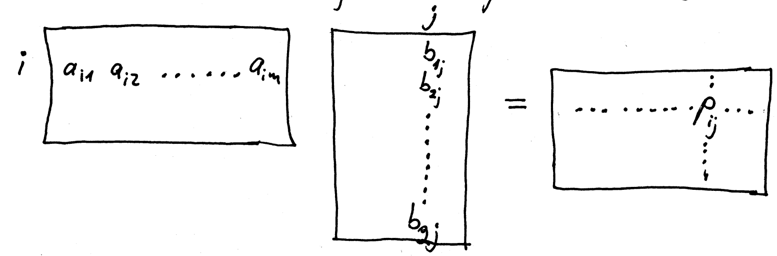
$$A \cdot B = [a_1 \; a_2 \dots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n.$$

npr. $[3 \;\; 1 \;\; 2] \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 3 - 1 + 8 = 10$

generalno:

$$[a_{ij}]_{m \times q} \cdot [b_{ij}]_{q \times s} = [p_{ij}]_{m \times s} \quad \text{gdje je} \quad p_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{im} b_{mj}$$

ovo znači proizvod i-tog reda od A i j-te kolone od B.



npr. $\begin{bmatrix} 0 & -1 & 2 \\ 3 & 4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Sistem linearnih jednačina

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m \end{aligned}$$

možemo pisati u matričnom obliku $A x = \mathbf{b}$, gdje A predstavlja koeficijent matricu $[a_{ij}]_{m \times n}$

$$\boxed{\quad A \quad} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Data je matrica $A = \begin{bmatrix} 1 & -2 & 5 \\ 3 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$. Izračunati

a) a_{11} b) a_{13} c) a_{31} d) $\sum_{i=1}^{3} a_{ii}$

R:

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 3 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

druga vrsta matrice A

prva kolona matrice A

Elementi matrice A su u obliku

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a) $a_{11} = 1$,

b) $a_{13} = 5$

c) $a_{31} = 2$

broj vrste

broj kolone

d) $\sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = 1 + (-2) + 1 = 0$

Posmatrajmo matricu B

$$B = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 0 & 1 & 2 \\ 2 & -1 & 4 & 1 \\ 0 & -3 & 1 & 3 \end{bmatrix}$$

Izračunati

a) b_{12} b) b_{21} c) b_{23} d) $\sum_{i=1}^{4} b_{ii}$

Posmatrajmo matrice

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 3 & -2 \\ 4 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 8 & 5 \\ 4 & -2 & 7 \\ 3 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \\ 2 & -4 \\ 5 & -2 \end{bmatrix}.$$

Ako postoji, izračunati:

(a) A^T (b) C^T (c) $A+B$

(d) $A+C$ (e) $(A+B)^T$ (f) A^T+B^T

(g) $B+B^T$ (h) $C+C^T$

Za matrice iz prethodnog zatka, ako postoje, izračunati sljedeće

(a) $A+A$ (b) $2A$

(c) $A+A+A$ (d) $4A+B$

\# Neka su $A=[a_{ij}]$ i $B=[b_{ij}]$ matrice iz $Mat_{4\times 3}(\mathbb{R})$ (skupa svih matrica oblika 4×3 čiji su elementi iz skupa realnih brojeva) definisane sa $a_{ij}=(-1)^{i+j}$ i $b_{ij}=i+j$.
Ako postoje, odrediti sljedeće matrice
(a) A^T (b) $A+B$ (c) A^T+B
(d) A^T+B^T (e) $(A+B)^T$ (f) $A+A$

Rj. Prvo odredimo šta su matrice A i B.

$a_{11}=(-1)^2=1$
$a_{12}=(-1)^3=-1$
$a_{13}=(-1)^4=1$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}_{4\times 3}$$

$b_{11}=1+1=2$
$b_{12}=1+2=3$
$b_{13}=1+3=4$

$$A=\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$$

a) $A^T=\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$,

(b) $A+B=\begin{bmatrix} 3 & 2 & 5 \\ 2 & 5 & 4 \\ 5 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}$

c) A^T+B ne postoji

d) $B^T=\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

završiti za vježbu

\# Neka su A i B matrice iz $Mat_{3\times 3}(\mathbb{R})$ definisane sa $A[i,j]=ij$ i $B[i,j]=i+j^2$.
(a) Izračunati $A+B$
(b) Izračunati $\sum_{i=1}^{3} A[i,i]$
(c) Da li je A jednaka svojoj transpoziciji A^T?
(d) Da li je B jednaka svojoj transpoziciji B^T?

\# Posmatrajmo matrice $A=\begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix}$ i $B=\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.
Izračunati sljedeće
(a) AB (b) BA (c) $A^2=A\cdot A$ d) B^2

\# (a) Za matrice iz prethodnog zadatka izračunati $(A+B)^2$ i $A^2+2AB+B^2$.
(b) Da li su rezultati dijela pod (a) isti? Diskutovati.

\# (a) Ispisati sve 3×3 matrice čiji su redovi $[1\ 0\ 0]$, $[0\ 1\ 0]$ i $[0\ 0\ 1]$.
(b) Koje od dobijenih matrica, djela pod (a), su jednake svojim transponovanim matricama.

U ovom zadatku A i B predstavljaju matrice. Da li su sljedeće tvrdnje tačne ili lažne?

(a) $(A^T)^T = A$ za sve A

(b) Ako je $A^T = B^T$ tada $A = B$.

(c) Ako je $A = A^T$, tada je A kvadratna matrica.

(d) Ako su A i B istog oblika, tada $(A+B)^T = A^T + B^T$.

Rj. Neka je $A_{m\times n}$ i $B_{p\times q}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \vdots & & & \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{bmatrix}$$

a)

$$(A^T)^T = \left(\begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}\right)^T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = A$$

prva tvrdnja je tačna

b)

završiti za vježbu

tvrdnje b), c) i d) su tačne

Za sve $n \in \mathbb{N}$, neka je $A_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$; $B_n = \begin{bmatrix} 1 & (-1)^n \\ -1 & 1 \end{bmatrix}$

(a) Odrediti A_n^T za sve $n \in \mathbb{N}$

(b) Izračunati $\{n \in \mathbb{N} : A_n^T = A_n\}$.

(c) Odrediti $\{n \in \mathbb{N} : B_n^T = B_n\}$

(d) Odrediti $\{n \in \mathbb{N} : B_n = B_0\}$

Rj.

a) $A_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, $A_n^T = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

b) $A_n^T = A_n$ akko $1=1$ $n=0$

$0=n$ $1=1$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

tj. $n=0$

$\{n \in \mathbb{N} : A_n^T = A_n\} = \emptyset$

c) $B_n = \begin{bmatrix} 1 & (-1)^n \\ -1 & 1 \end{bmatrix}$, $B_n^T = \begin{bmatrix} 1 & -1 \\ (-1)^n & 1 \end{bmatrix}$

$B_n^T = B_n$ akko $1=1$ $(-1)^n = -1$

$(-1) = (-1)^n$ $1=1$

tj. $n = 1, 3, 5, 7, \dots$

$\{n \in \mathbb{N} : B_n^T = B_n\} = \{1, 2, 3, 5\ 7, \dots\}$ ← svi nepar. prirodn. brojevi

(d) za vježbu

(#) Neka je $A=\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}$. Izračunati A^7.

Rj:

$$A^2=A\cdot A=\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}=\begin{bmatrix}1&1&0\\0&0&1\\1&0&1\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}1&1&0\\0&0&1\\1&0&1\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}=\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}$$

$$A^4=A^3\cdot A=\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}=\begin{bmatrix}1&1&1\\1&0&1\\2&1&1\end{bmatrix}$$

$$A^7=A^4\cdot A^3=\begin{bmatrix}1&1&1\\1&0&1\\2&1&1\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}3&2&2\\2&1&2\\4&2&3\end{bmatrix}$$

$$A^7=\begin{bmatrix}3&2&2\\2&1&2\\4&2&3\end{bmatrix}$$

1. Ako je $A=\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}$ i $B=\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}$ izračunati:

a) $A+B$ b) $A-B$ c) $2A-3B-I$ (I jedinična matrica)

Rj: a) $\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}+\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}=\begin{bmatrix}3&3&11\\6&2&10\\6&3&17\end{bmatrix}$ b) $\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}-\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}=\begin{bmatrix}1&5&-1\\0&2&2\\-4&-1&-3\end{bmatrix}$

c) $2\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}-3\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}4&8&10\\6&4&12\\2&2&14\end{bmatrix}-\begin{bmatrix}3&-3&18\\9&0&12\\15&6&30\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&11&-8\\-3&3&0\\-13&-4&-17\end{bmatrix}$

2. Izračunati:

a) $\begin{bmatrix}2&3\\1&6\\0&1\end{bmatrix}\cdot\begin{bmatrix}2&1\\3&5\end{bmatrix}$ Rj: $=\begin{bmatrix}2\cdot2+3\cdot3&2\cdot1+3\cdot5\\1\cdot2+6\cdot3&1\cdot1+6\cdot5\\0\cdot2+1\cdot3&0\cdot1+1\cdot5\end{bmatrix}=\begin{bmatrix}13&17\\20&31\\3&5\end{bmatrix}$

b) $\begin{bmatrix}1&4\\2&-5\\3&6\end{bmatrix}\begin{bmatrix}1&4&-2\\2&5&6\end{bmatrix}=\begin{bmatrix}1\cdot1+4\cdot2&1\cdot4+4\cdot5&1\cdot(-2)+4\cdot6\\2\cdot1+(-5)\cdot2&2\cdot4+(-5)\cdot5&2\cdot(-2)+(-5)6\\3\cdot1+6\cdot2&3\cdot4+6\cdot5&3\cdot(-2)+6\cdot6\end{bmatrix}=\begin{bmatrix}9&24&22\\-8&-17&-34\\15&42&30\end{bmatrix}$

c) $\begin{bmatrix}2\\1\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}=\begin{bmatrix}2&4&6\\1&2&3\\3&6&9\end{bmatrix}$ d) $\begin{bmatrix}a&b&c\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=a+2b+3c$

3. Ako je $A=\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}$ izračunati $3A^2-2A^T+5\cdot I$.

(A^T transponovana matrica matrice A) (kada elementi iz reda zamjene položaj sa elementima iz kolona)

Rj: $A^T=\begin{bmatrix}1&2&3\\-2&-4&-5\\3&1&2\end{bmatrix}$. $A^2=\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}=\begin{bmatrix}6&-9&7\\-3&7&4\\-1&4&8\end{bmatrix}$

$$3A^2-2A^T+5=\begin{bmatrix}18&-27&21\\-9&21&12\\-3&12&24\end{bmatrix}-\begin{bmatrix}2&4&6\\-4&-8&-10\\6&2&4\end{bmatrix}+\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=\begin{bmatrix}21&-31&15\\-5&34&22\\-9&10&25\end{bmatrix}$$

4. Ako je $A=\begin{bmatrix}2&3&5\\-3&1&5\end{bmatrix}$ i $B=\begin{bmatrix}-2&-3\\-1&0\\1&1\end{bmatrix}$, izračunati $2\cdot A^T\cdot A-3\cdot B\cdot B^T+6I$.

Rj: $\begin{bmatrix}-7&0&5\\0&23&43\\5&43&100\end{bmatrix}$

Determinante

matrica tipa $n \times n$

Determinanta je broj pridružen svakoj kvadratnoj matrici;
Determinantu matrice A obilježavamo sa $\det A$ ili $|A|$.
Preciznija definicija determinante je:
Determinanta je f-ja koja $n \times n$ realnih brojeva preslikava u realan broj.
Osobine determinante: (neke osobine determinanti)

1. Determinanta jedinične matrice je 1 ($\det I = 1$).
2. Ako dva reda (ili dvije kolone) međusobno zamjene mjesta znak determinante se mijenja.

$$\begin{vmatrix}1&0\\0&1\end{vmatrix}=1, \quad \begin{vmatrix}0&1\\1&0\end{vmatrix}=-1, \quad \begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$$

3. a) Determinanta se množi jednim brojem ako se tim brojem pomnože svi elementi jednog reda (ili jedne kolone)

$$t\begin{vmatrix}a&b\\c&d\end{vmatrix}=\begin{vmatrix}ta&tb\\c&d\end{vmatrix}$$

b) $$\begin{vmatrix}a+a'&b+b'\\c&d\end{vmatrix}=\begin{vmatrix}a&b\\c&d\end{vmatrix}+\begin{vmatrix}a'&b'\\c&d\end{vmatrix}$$

(linearnost za svaki red)

1. Izračunati: $\begin{vmatrix}+&-&+\\-&+&-\\+&-&+\end{vmatrix}$

a) $$\begin{vmatrix}1&1&0\\0&1&1\\2&0&0\end{vmatrix} \overset{Rj.}{=} 2\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}-0\cdot\begin{vmatrix}1&0\\0&1\end{vmatrix}+0\cdot\begin{vmatrix}1&1\\0&1\end{vmatrix}=2\cdot1=2$$

razvoj determinante po trećem redu

b) $$\begin{vmatrix}1&2&0\\0&1&1\\3&1&1\end{vmatrix} \underset{\text{po prvom redu}}{\overset{\text{razvoj determinante}}{=}} 1\cdot\begin{vmatrix}1&1\\1&1\end{vmatrix}-2\cdot\begin{vmatrix}0&1\\3&1\end{vmatrix}+0\cdot\begin{vmatrix}0&1\\3&1\end{vmatrix}$$

$$=1\cdot0-2\cdot(-3)+0=6$$

Mogli smo izračunati i na sljedeći način:

$$\begin{vmatrix}1&2&0\\0&1&1\\3&1&1\end{vmatrix} \overset{II_k-III_k}{=} \begin{vmatrix}1&2&0\\0&0&1\\3&0&1\end{vmatrix}=(-2)\cdot\begin{vmatrix}0&1\\3&1\end{vmatrix}=(-2)\cdot(-3)=6$$

2. Izračunati:

a) $$\begin{vmatrix}1&0&1&0\\1&1&0&0\\2&0&1&1\\0&0&1&1\end{vmatrix} \overset{III_k-IV_k}{=} \begin{vmatrix}1&0&1&0\\1&1&0&0\\2&0&0&1\\0&0&0&1\end{vmatrix}=1\cdot\begin{vmatrix}1&1&0\\2&0&1\\0&0&1\end{vmatrix}=(-1)\begin{vmatrix}2&1\\0&1\end{vmatrix}=-2$$

b) $$\begin{vmatrix}4&1&0&3\\2&0&1&1\\4&1&0&1\\0&1&0&3\end{vmatrix} \overset{I_R-IV_R}{=} \begin{vmatrix}4&0&0&0\\2&0&1&1\\4&1&0&1\\0&1&0&3\end{vmatrix}=4\begin{vmatrix}0&1&1\\1&0&1\\1&0&3\end{vmatrix}=4\cdot(-1)\begin{vmatrix}1&1\\1&3\end{vmatrix}=(-4)\cdot2=-8$$

3. Izračunati:

a) $$\begin{vmatrix}3&-2&1\\4&-1&1\\1&1&5\end{vmatrix} \overset{Rj.\ I_R-II_R}{=} \begin{vmatrix}-1&-1&0\\4&-1&1\\1&1&5\end{vmatrix} \overset{III_R+I_R}{=} \begin{vmatrix}-1&-1&0\\4&-1&1\\0&0&5\end{vmatrix}=5\begin{vmatrix}-1&-1\\4&-1\end{vmatrix}$$

$$=5\cdot5=25$$

b) $$\begin{vmatrix}1&3&3\\2&-1&4\\1&2&7\end{vmatrix} \overset{I_R-III_R}{=} \begin{vmatrix}0&1&-4\\2&-1&4\\1&2&7\end{vmatrix} \overset{II_R+I_R}{=} \begin{vmatrix}0&-1&-4\\2&0&0\\1&2&7\end{vmatrix}=(-2)\cdot\begin{vmatrix}1&-4\\2&7\end{vmatrix}$$

$$=(-2)\cdot15=-30$$

4. Izračunati:

a) $$\begin{vmatrix}1&0&1&0\\2&5&2&0\\3&0&0&1\\4&3&2&1\end{vmatrix} \underset{\substack{III_R-I_R\cdot3\\ IV_R-I_R\cdot4}}{\overset{Rj.\ II_R-I_R\cdot2}{=}} \begin{vmatrix}1&0&1&0\\0&5&0&0\\0&0&-3&1\\0&3&-2&1\end{vmatrix}=1\cdot\begin{vmatrix}5&0&0\\0&-3&1\\3&-2&1\end{vmatrix}=5\cdot\begin{vmatrix}-3&1\\-2&1\end{vmatrix}$$

$$=5\cdot(-1)=-5$$

b)^v $\begin{vmatrix}0&0&1&2\\1&2&0&0\\1&2&3&4\\0&0&1&5\end{vmatrix}$ c)^v $\begin{vmatrix}5&4&3&2\\1&1&2&4\\4&3&2&1\\2&2&2&1\end{vmatrix}$

Rješenja:
b) 0 c) -1

5.^{v} Izračunati: $\begin{vmatrix}\sqrt3&2\sqrt2&\sqrt5\\5\sqrt3&\sqrt8&7\sqrt5\\\sqrt5+2\sqrt3&4\sqrt2&\sqrt3+2\sqrt5\end{vmatrix}$.

Rj. $36\sqrt2$

(6.) Dokazati da je $\begin{vmatrix}1 & a & a^2+a^3\\ 1 & a^2 & a^3+a\\ 1 & a^3 & a+a^2\end{vmatrix}=0$.

Rj:

$$\begin{vmatrix}1&a&a^2+a^3\\ 1&a^2&a^3+a\\ 1&a^3&a+a^2\end{vmatrix} = a\begin{vmatrix}1&1&a^2(1+a)\\ 1&a&a(a^2+1)\\ 1&a^2&a(1+a)\end{vmatrix} = a\cdot a\cdot\begin{vmatrix}1&1&a(a+1)\\ 1&a&a^2+1\\ 1&a^2&a+1\end{vmatrix} \underset{III_R-I_R}{\overset{II_R-I_R}{=\!=}}$$

$$= a^2\begin{vmatrix}1&1&a(a+1)\\ 0&a-1&1-a\\ 0&a^2-1&1-a^2\end{vmatrix} = a^2\begin{vmatrix}a-1&1-a\\ (a+1)(a-1)&1-a^2\end{vmatrix} = a^2(a-1)\begin{vmatrix}1&1-a\\ a+1&(1-a)(1+a)\end{vmatrix}$$

$$= a^2(a-1)(1-a)\begin{vmatrix}1&1\\ a+1&a+1\end{vmatrix} = a^2(a-1)(1-a)(a+1)\underbrace{\begin{vmatrix}1&1\\ 1&1\end{vmatrix}}_{=0} = 0$$

što je i trebalo dobiti.

(7.) Izračunati: $\begin{vmatrix}a&b&a&b\\ b&a&a&b\\ a&b&b&a\\ b&a&b&a\end{vmatrix}$

Rj. $\overset{IV_K+(I_K+II_K+III_K)}{=\!=}\begin{vmatrix}a&b&a&2a+2b\\ b&a&a&2a+2b\\ a&b&b&2a+2b\\ b&a&b&2a+2b\end{vmatrix}$

$$=(2a+2b)\begin{vmatrix}a&b&a&1\\ b&a&a&1\\ a&b&b&1\\ b&a&b&1\end{vmatrix} \underset{IV_R-III_R}{\overset{II_R-I_R}{=\!=}}(2a+2b)\begin{vmatrix}a&b&a&1\\ b-a&a-b&0&0\\ a&b&b&1\\ b-a&a-b&0&0\end{vmatrix}\overset{III_R-I_R}{=\!=}(2a+2b)\cdot$$

$$\cdot\begin{vmatrix}a&b&a&1\\ b-a&a-b&0&0\\ 0&0&b-a&1\\ b-a&a-b&0&0\end{vmatrix} = -(2a+2b)\cdot\begin{vmatrix}a&b&a\\ b-a&a-b&0\\ b-a&a-b&0\end{vmatrix} = -a(2a+2b)\begin{vmatrix}b-a&a-b\\ b-a&a-b\end{vmatrix}$$

$$= -a(2a+2b)(b-a)(a-b)\begin{vmatrix}1&1\\ 1&1\end{vmatrix}=0.$$

(8.) Rastaviti na faktore polinom:

a) $\begin{vmatrix}1&1&1\\ x^2&y^2&z^2\\ x^3&y^3&z^3\end{vmatrix}$ b) $\begin{vmatrix}a&b&a+b\\ b&a+b&a\\ a+b&a&b\end{vmatrix}$ c) $\begin{vmatrix}a^2&(a+1)^2&(a+2)^2\\ b^2&(b+1)^2&(b+2)^2\\ c^2&(c+1)^2&(c+2)^2\end{vmatrix}$.

(#) Riješiti jednačinu $\begin{vmatrix}3x-5&-5-2x&x+1\\ 2x-4&-2-2x&x-1\\ 3x-8&2-3x&2x-5\end{vmatrix}=0$.

Rj.

$$\begin{vmatrix}3x-5&-5-2x&x+1\\ 2x-4&-2-2x&x-1\\ 3x-8&2-3x&2x-5\end{vmatrix}=(-1)\begin{vmatrix}3x-5&2x+5&x+1\\ 2x-4&2x+2&x-1\\ 3x-8&3x-2&2x-5\end{vmatrix}\overset{III_V-II_V}{=\!=}$$

$$\begin{vmatrix}3x-5&2x+5&x+1\\ 2x-4&2x+2&x-1\\ x-4&x-4&x-4\end{vmatrix}=(-1)(x-4)\begin{vmatrix}3x-5&2x+5&x+1\\ 2x-4&2x+2&x-1\\ 1&1&1\end{vmatrix}\underset{II_K-III_K}{\overset{I_K-III_K}{=\!=}}$$

$$=(-1)(x-4)\begin{vmatrix}2x-6&x+4&x+1\\ x-3&x+3&x-1\\ 0&0&1\end{vmatrix}=(-1)(x-4)\begin{vmatrix}2x-6&x+4\\ x-3&x+3\end{vmatrix}\overset{I_V-II_V}{=\!=}$$

$$=(-1)(x-4)\begin{vmatrix}x-3&1\\ x-3&x+3\end{vmatrix}=(-1)(x-4)(x-3)\begin{vmatrix}1&1\\ 1&x+3\end{vmatrix}=(-1)(x-4)(x-3)(x+2)$$

$(-1)(x-4)(x-3)(x+2)=0$ Rješenja jednačine su $x=4$ ili $x=3$ ili $x=-2$.

(#) Riješiti jednačinu: $\begin{vmatrix}x-3&x+2&x-1\\ x+2&x-4&x\\ x-1&x+4&x-5\end{vmatrix}=0$.

Rj.

$$\begin{vmatrix}x-3&x+2&x-1\\ x+2&x-4&x\\ x-1&x+4&x-5\end{vmatrix}\overset{I_K+II_K+III_K}{=\!=}\begin{vmatrix}3x-2&x+2&x-1\\ 3x-2&x-4&x\\ 3x-2&x+4&x-5\end{vmatrix}=(3x-2)\begin{vmatrix}1&x+2&x-1\\ 1&x-4&x\\ 1&x+4&x-5\end{vmatrix}$$

$$\underset{III_R-II_R}{\overset{I_R-II_R}{=\!=}}(3x-2)\begin{vmatrix}0&6&-1\\ 1&x-4&x\\ 0&8&-5\end{vmatrix}=-(3x-2)\begin{vmatrix}6&-1\\ 8&-5\end{vmatrix}=-(3x-2)(-30+8)=$$

$= 22(3x-2)$

$22(3x-2)=0$

$3x-2=0$

$3x=2$

$x=\frac{2}{3}$ je rješenje jednačine

Izračunati

$$\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix}.$$

Rj.

$$\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix} \xlongequal[III_k + IV_k \cdot 2]{I_k + III_k,\ II_k + III_k} \begin{vmatrix} 4 & a+3 & 7 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ -2 & -2 & -3 & 1 \\ -3 & -3 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 4 & a+3 & 7 \\ -2 & -2 & -3 \\ -3 & -3 & 3 \end{vmatrix} \xlongequal[II_k + III_k]{I_k + III_k}$$

$$= \begin{vmatrix} 11 & a+10 & 7 \\ -5 & -5 & -3 \\ 0 & 0 & 3 \end{vmatrix} = 3 \begin{vmatrix} 11 & a+10 \\ -5 & -5 \end{vmatrix} = 3\cdot(-5) \begin{vmatrix} 11 & a+10 \\ 1 & 1 \end{vmatrix} = -15(11-a-10)$$

$$= -15(-a+1) = 15a - 15$$

Rastaviti na faktore polinome:

a) $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ b) $\begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix}.$

Rj.

a) $$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} \xlongequal[II_k - III_k]{I_k - III_k} \begin{vmatrix} 0 & 0 & 1 \\ x^2-z^2 & y^2-z^2 & z^2 \\ x^3-z^3 & y^3-z^3 & z^3 \end{vmatrix} =$$

$$= 1\cdot \begin{vmatrix} (x-z)(x+z) & (y-z)(y+z) \\ (x-z)(x^2+xz+z^2) & (y-z)(y^2+yz+z^2) \end{vmatrix} = (x-z)(y-z) \begin{vmatrix} x+z & y+z \\ x^2+xz+z^2 & y^2+yz+z^2 \end{vmatrix}$$

$$\xlongequal{I_k - II_k} (x-z)(y-z) \begin{vmatrix} x-y & y+z \\ x^2-y^2+xz-yz & y^2+yz+z^2 \end{vmatrix} = (x-z)(y-z) \begin{vmatrix} x-y & y+z \\ (x-y)(x+y)+z(x-y) & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-z)(y-z)(x-y) \begin{vmatrix} 1 & y+z \\ x+y+z & y^2+yz+z^2 \end{vmatrix} = (x-z)(y-z)(x-y)\big(y^2 + \cancel{yz} + z^2 - xy - y^2 - \cancel{yz}$$

$$- xz - yz - z^2\big) = (x-z)(y-z)(x-y)(-xy-xz-yz)$$

b) $$\begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix} \xlongequal{I_k + (II_k + III_k)} \begin{vmatrix} 2a+2b & b & a+b \\ 2a+2b & a+b & a \\ 2a+2b & a & b \end{vmatrix} = 2(a+b) \begin{vmatrix} 1 & b & a+b \\ 1 & a+b & a \\ 1 & a & b \end{vmatrix}$$

$$\xlongequal[III_v - II_v]{I_v - II_v} 2(a+b) \begin{vmatrix} 0 & -a & b \\ 1 & a+b & a \\ 0 & -b & b-a \end{vmatrix} = 2(a+b)(-1) \begin{vmatrix} -a & b \\ -b & b-a \end{vmatrix} = 2(a+b) \begin{vmatrix} a & b \\ b & b-a \end{vmatrix}$$

$$= 2(a+b)(ab-a^2-b^2) = -2(a+b)(a^2-ab+b^2) = -2(a^3+b^3)$$

Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1+x^2 & x & 0 & \dots & 0 & 0\\ x & 1+x^2 & x & \dots & 0 & 0 \\ 0 & x & 1+x^2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1+x^2 & x\\ 0 & 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2n}$$

(determinanta ima n vrsta i n kolona).

Rj. BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2

$$\begin{vmatrix} 1+x^2 & x\\ x & 1+x^2 \end{vmatrix} = (1+x^2)^2 - x^2 = 1+2x^2+x^4-x^2 = 1+x^2+x^4$$

Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost tačna za determinantu koja ima k vrsta i k kolona

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0\\ x & 1+x^2 & \dots & 0 & 0\\ \vdots & \vdots & & & \vdots\\ 0 & 0 & \dots & 1+x^2 & x\\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2k}$$

gdje k uzima brojeve od 1 do n.

Na osnovu ove pretpostavke dokažimo da je jednakost tačna za determinantu koja ima n+1 vrstu i n+1 kolonu tj. dokažimo da

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0\\ x & 1+x^2 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & 1+x^2 & x\\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2n}+x^{2n+2}.$$

Polazimo od determinante koja ima (n+1)-vrstu i (n+1)-kolonu:

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0\\ x & 1+x^2 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & 1+x^2 & x\\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} \underset{\text{koloni}}{\overset{\text{razvoj po prvoj}}{=}} (1+x^2)\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0\\ x & 1+x^2 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & 1+x^2 & x\\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} - x\begin{vmatrix} x & 0 & 0 & \dots & 0 & 0\\ x & 1+x^2 & x & \dots & 0 & 0\\ 0 & x & 1+x^2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1+x^2 & x\\ 0 & 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} =$$

(ova determinanta ima n vrsta i n kolona)

(ovu determinantu mogu razviti po prvoj vrsti i ostaće mi determinanta iz pretpostavke koja ima n-1 vrstu i n-1 kolonu)

$$\underset{\text{pretpostavke}}{\overset{\text{na osnovu}}{=}} (1+x^2)(1+x^2+x^4+\dots+x^{2n}) - x^2(1+x^2+x^4+\dots+x^{2n-2})$$

$$= (1+x^2+x^4+\dots+x^{2n}) + (x^2+x^4+x^6+\dots+x^{2n}+x^{2n+2}) - (x^2+x^4+x^6+\dots+x^{2n-2}+x^{2n}) = 1+x^2+x^4+\dots+x^{2n+2}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve

Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1 & n & n & \dots & n & n\\ n & 2 & n & \dots & n & n\\ n & n & 3 & \dots & n & n\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ n & n & n & \dots & n-1 & n\\ n & n & n & \dots & n & n \end{vmatrix} = (-1)^{n-1}\cdot n!$$

Rj. BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2.

$$\begin{vmatrix} 1 & 2\\ 2 & 2 \end{vmatrix} = 2-4 = -2 = (-1)^{2-1}\cdot 2!$$

Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost

$$\begin{vmatrix} 1 & k & k & \dots & k & k\\ k & 2 & k & \dots & k & k\\ k & k & 3 & \dots & k & k\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ k & k & k & \dots & k-1 & k\\ k & k & k & \dots & k & k \end{vmatrix} = (-1)^{k-1}\cdot k!$$

tačna za sve brojeve od 1 do n $(k=1,2,\dots,n)$.

Uz pomoć ove pretpostavke dokažimo da je jednakost tačna za broj n+1 tj. dokažimo

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1\\ n+1 & 2 & \dots & n+1 & n+1\\ \vdots & \vdots & & \vdots & \vdots\\ n+1 & n+1 & \dots & n & n+1\\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-1)^n\cdot(n+1)!$$

ZAKLJUČAK
Jednakost je tačna za sve prirodne brojeve

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1\\ n+1 & 2 & \dots & n+1 & n+1\\ \vdots & \vdots & & \vdots & \vdots\\ n+1 & n+1 & \dots & n & n+1\\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} \underline{\underline{I_k-(N+1)_k}} \begin{vmatrix} -n & n+1 & \dots & n+1 & n+1\\ 0 & 2 & \dots & n+1 & n+1\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & n+1 & \dots & n & n+1\\ 0 & n+1 & \dots & n+1 & n+1 \end{vmatrix} =$$

$$= (-n)\begin{vmatrix} 2 & n+1 & \dots & n+1 & n+1\\ n+1 & 3 & \dots & n+1 & n+1\\ \vdots & \vdots & & \vdots & \vdots\\ n+1 & n+1 & \dots & n & n+1\\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-n)(n+1)\begin{vmatrix} 2 & n+1 & \dots & n+1 & 1\\ n+1 & 3 & \dots & n+1 & 1\\ \vdots & \vdots & & \vdots & \vdots\\ n+1 & n+1 & \dots & n & 1\\ n+1 & n+1 & \dots & n+1 & 1 \end{vmatrix} \begin{array}{l} I_k - N_k\\ II_k - N_k\\ \vdots\\ (N-1)_k - N_k \end{array}$$

$$= (-1)\cdot n(n+1)\begin{vmatrix} 1 & n & \dots & n & 1\\ n & 2 & \dots & n & 1\\ \vdots & \vdots & & & \vdots\\ n & n & \dots & n-1 & 1\\ n & n & \dots & n & 1 \end{vmatrix} = (-1)(n+1)\begin{vmatrix} 1 & n & \dots & n & n\\ n & 2 & \dots & n & n\\ \vdots & \vdots & & \vdots & \vdots\\ n & n & \dots & n-1 & n\\ n & n & \dots & n & n \end{vmatrix} \underset{\text{pretpostavke}}{\overset{\text{na osnovu}}{=}} (-1)(n+1)(-1)^{n-1}\cdot n! = (-1)^n(n+1)!$$

Rang matrice

Dijagonalne i trougaone matrice

a) Matrice oblike $D = \begin{bmatrix} \lambda_0 & 0 & ... & 0 \\ 0 & \lambda_1 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda_d \end{bmatrix}$ zovemo *dijagonalne matrice* i često ih označavamo sa $diag(\lambda_0, \lambda_1, ..., \lambda_d)$.

b) *Glavna dijagonala* kvadratne matrice su elementi koji se nalaze na dijagonalnoj liniji koja počinje u gornjem lijevom uglu matrice a završava u donjem desnom uglu. Za kvadratnu matricu kažemo da je *trougaona matrica* ako su svi elementi iznad glavne dijagonale ili ispod glavne dijagonale jednaki nula. Za kavadratnu matricu kažemo da je *gornje-trougaona matrica* ako su svi elementi ispod glavne dijagonale jednaki nula. Za kvadratnu matricu kažemo da je *donje-trougaona matrica* ako su svi elementi iznad glavne dijagonale jednaki nuli. $\diamond$

Inverzna matrica

Za datu kvadratnu matricu $A_{n\times n}$, matricu $B_{n\times n}$ koja zadovoljava uslov

$$AB = I \qquad \text{i} \qquad BA = I$$

zovemo *inverz* od A i označavamo sa $B = A^{-1}$. Nisu sve kvadratne matrice invertibilne - nula matrica je trivijalni primjer, i postoji veliki broj nenula matrica koje nisu invertibilne. Za invertibilnu matricu kažemo da je *nesingularna*, a za kvadratnu matricu koja nema inverznu matricu kažemo da je *singularna matrica*. $\diamond$

Saglasan i nesaglasan sistem

Za sistem od m linearnih jednačina sa n nepoznatih kažemo da je *saglasan sistem* ako posjeduje bar jedno rješenje. Ako sistem nema rješenja, tada za sistem kažemo da je *nesaglasan sistem*. $\diamond$

Elementarne red (kolona) operacije

Elementarne red (kolona) operacije su:

(i) Zamjena mjesta redova (kolona) i i j.

(ii) Množenje reda (kolone) i sa $\alpha \neq 0$.

(iii) Dodavanje reda (kolone) i pomnožene nekim brojem redu (koloni) j. $\diamond$

Ekvivalencija

(i) Kadgod matricu B možemo dobiti iz matrice A kombinacijom elementarnih red ili kolona operacija, pišemo $A \sim B$, i kažemo da su A i B *ekvivalentne* matrice. S obzirom da su elementarne red i kolona operacije u stvari množenje redom sa lijeve i desne strane elementarnim matricama može se dokazati da

$$A \sim B \quad \Leftrightarrow \quad PAQ = B \text{ za nesingularne } P \text{ i } Q$$

(ii) Kadgod se matrica B može dobiti iz matrice A primjenjujući samo red operacije, pišemo $A \overset{red}{\sim} B$, i kažemo da su matrice A i B *red ekvivalentne*. Drugim riječima

$$A \overset{red}{\sim} B \quad \Leftrightarrow \quad PA = B \text{ za nesingularnu } P.$$

(iii) Kad god se matrica B može dobiti iz matrice A primjenjujući samo niz uzastopnih kolona operacija, pišemo $A \overset{kol}{\sim} B$, i kažemo da su matrice A i B *kolona ekvivalentne*. Drugim riječima

$$A \overset{kol}{\sim} B \quad \Leftrightarrow \quad AQ = B \text{ za nesingularnu } Q.$$

$\diamond$

Red ešelon oblik

Za $m \times n$ matricu E, sa redovima E_{i*} i kolonama E_{*j}, kažemo da je u red ešelon obliku ako sljedeća dva uslova vrijede:

(a) Ako su svi elementi reda E_{i*} jenaki nuli, tada su i svi elementi u redovima ispod E_{i*} jednaki nuli, tj. svi nula redovi su na dnu matrice.

(b) Ako se prvi nenula elemenat u E_{i*} nalazi na $j$toj poziciji, tada su svi elementi ispod i-te pozicije u kolonama E_{*1}, E_{*2},...,E_{*j} nule.

Ova dva uslova kažu da nenula elementi u ešelon obliku moraju ležati na ili iznad glavne linije stepenica čiji je početak u gornjem lijevom uglu matrice i postepeno pada prema dole desno. Pivoti su prvi nenula elementi u ešelon redovima. Tipična struktura za matricu koja je u red ešelon obliku je ilustrirana ispod, gdje su pivoti zaokruženi.

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\diamond$

Rang matrice

Pretpostavimo da je matrica $A_{m\times n}$ pomoću red operacija svedena na red ešelon oblik E. Rang matrice A se definiše kao broj

$$
\begin{aligned}
rang(A) &= \text{broj pivota}\\
&= \text{broj nenula redova u } E\\
&= \text{broj osnovnih kolona u } A
\end{aligned}
$$

gdje su *osnovne kolone* od A definisane kao one kolone u A koje sadrže pivot pozicije. $\diamond$

Reducirani red ešelon oblik

Za matricu $E_{m\times n}$ kažemo da je u *reduciranom red ešelon obliku* ako su sljedeća tri uslova ispunjena.

(i) E je u red ešelon obliku.

(ii) Prvi nenula elemenat u svakom redu (tj. svaki pivot) je 1.

(iii) Sve vrijednosti iznad svakog pivota su 0.

Tipična struktura za matricu u reduciranom red ešelon obliku je ilustrirana ispod, gdje elementi označeni sa $*$ mogu biti ili nula ili nenula brojevi:

$$
\begin{pmatrix}
1 & * & 0 & 0 & * & * & 0 & *\\
0 & 0 & 1 & 0 & * & * & 0 & *\\
0 & 0 & 0 & 1 & * & * & 0 & *\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & *\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

$\diamond$

1. Odrediti rang matrice:

a) $M=\begin{bmatrix}2 & -1 & 3 & -2 & 4\\ 4 & -2 & 4 & 1 & 7\\ 2 & -1 & 1 & 8 & 2\end{bmatrix}$. Rj. $\overset{II_R-I_R\cdot 2}{\underset{III_R-I_R}{\sim}} \begin{bmatrix}2 & -1 & 3 & -2 & 4\\ 0 & 0 & -2 & 5 & -1\\ 0 & 0 & -2 & 10 & -2\end{bmatrix} \overset{III_R-II_R}{\sim} \begin{bmatrix}2 & -1 & 3 & -2 & 4\\ 0 & 0 & -2 & 5 & -1\\ 0 & 0 & 0 & 5 & -1\end{bmatrix}$, $rang(M)=3$

b) $A=\begin{bmatrix}-2 & 1 & 0 & 2\\ 0 & -1 & 1 & 3\\ -1 & 1 & 0 & -2\\ -4 & 2 & 1 & 1\end{bmatrix}$ Rj. $\overset{I_k\leftrightarrow II_k}{\sim} \begin{bmatrix}1 & -2 & 0 & 2\\ -1 & 0 & 1 & 3\\ 1 & -1 & 0 & -2\\ 2 & -4 & 1 & 1\end{bmatrix} \overset{II_R+I_R}{\underset{\substack{III_R-I_R\\ IV_R-I_R\cdot 2}}{\sim}} \begin{bmatrix}1 & -2 & 0 & 2\\ 0 & -2 & 1 & 5\\ 0 & 1 & 0 & -4\\ 0 & 0 & 1 & -3\end{bmatrix} \overset{II_k\leftrightarrow III_k}{\sim}$

$\sim \begin{bmatrix}1 & 0 & -2 & 2\\ 0 & 1 & -2 & 5\\ 0 & 0 & 1 & -4\\ 0 & 1 & 0 & -3\end{bmatrix} \overset{IV_R-I_R}{\sim} \begin{bmatrix}1 & 0 & -2 & 2\\ 0 & 1 & -2 & 5\\ 0 & 0 & 1 & -4\\ 0 & 0 & 2 & -8\end{bmatrix} \overset{IV_R-III_R\cdot 2}{\sim} \begin{bmatrix}1 & 0 & -2 & 2\\ 0 & 1 & -2 & 5\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0\end{bmatrix}$, $rang(A)=2$

2. Odrediti rang matrice $A=\begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 3 & -1 & 1\\ 3 & 4 & 0 & \lambda+2\end{bmatrix}$, $\lambda\in\mathbb{R}$.

Rj. $\begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 3 & -1 & 1\\ 3 & 4 & 0 & \lambda+2\end{bmatrix} \overset{III_k\leftrightarrow I_k}{\sim} \begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 3 & 2 & 1\\ 0 & 4 & 3 & \lambda+2\end{bmatrix} \overset{II_V+I_V}{\sim} \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 4 & 3 & 2\\ 0 & 4 & 3 & \lambda+2\end{bmatrix} \overset{III_V-II_V}{\sim}$

$\sim \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 4 & 3 & 2\\ 0 & 0 & 0 & \lambda\end{bmatrix}$, ako je $\lambda=0$ tada je $rang(A)=2$

ako je $\lambda\neq 0$ tada je $rang(A)=3$

3. U ovisnosti o parametru $\lambda\in\mathbb{R}$ odredite rang matrice

$A=\begin{bmatrix}1 & 1 & 1\\ 1 & \lambda & \lambda^2\\ 1 & \lambda^2 & \lambda\end{bmatrix}$.

Rj. $\begin{bmatrix}1 & 1 & 1\\ 1 & \lambda & \lambda^2\\ 1 & \lambda^2 & \lambda\end{bmatrix} \overset{II_R-I_R}{\underset{III_R-I_R}{\sim}} \begin{bmatrix}1 & 1 & 1\\ 0 & \lambda-1 & \lambda^2-1\\ 0 & \lambda^2-1 & \lambda-1\end{bmatrix} \overset{II_R:(\lambda-1)}{\underset{\substack{III_R:(\lambda-1)\\ \lambda\neq 1\\ \text{zapamtimo}}}{\sim}} \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & \lambda+1\\ 0 & \lambda+1 & 1\end{bmatrix} \overset{III_R+II_R\cdot(-\lambda-1)}{\sim} \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & \lambda+1\\ 0 & 0 & \underbrace{-(\lambda+1)^2+1}_{-(\lambda^2+2\lambda+1)+1}\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & \lambda+1\\ 0 & 0 & -\lambda(\lambda+2)\end{bmatrix}$

Matrica se ne može više pojednostaviti. Diskusija:

Za $\lambda=0$ dobićemo $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} \Rightarrow rang\,A=2$

Za $\lambda=-2$ imamo $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix} \Rightarrow rang\,A=2$.

Ostaje nam još slučaj $\lambda=1$. Zašto?

Za $\lambda=1$, $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \Rightarrow rang\,A=1$. Zašto?

U ostalim slučajevima (tj. kad je $\lambda\neq 0$, $\lambda\neq -2$ i $\lambda\neq 1$) $rang\,A=3$.

4. Diskutovati rang matrice $M=\begin{bmatrix}1 & 10 & -6 & \lambda\\ 2 & -1 & \lambda & 3\\ 1 & \lambda & -1 & 2\end{bmatrix}$.

5. Diskutovati o rangu matrice

$M=\begin{bmatrix}a & b & 1\\ 1 & ab & 1\\ 1 & b & a\end{bmatrix}$ u zavisnosti od parametara a i b.

Diskutovati rang matrice u zavisnosti od parametara a i b,

$$A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix}$$

Rj.

$$A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix} \overset{II_k \leftrightarrow IV_k}{\sim} \begin{bmatrix} 2 & 6 & 9 & 3 & 2 \\ 5 & 8 & 12 & 4 & 5 \\ 1 & 4 & 6 & 2 & 1 \\ 4 & 2 & 3 & 1 & a \\ 3 & 4 & 6 & b & 3 \\ 7 & 10 & 15 & 5 & 7 \end{bmatrix} \overset{IV_R \leftrightarrow VI_R}{\sim} \begin{bmatrix} 2 & 6 & 9 & 3 & 2 \\ 5 & 8 & 12 & 4 & 5 \\ 1 & 4 & 6 & 2 & 1 \\ 7 & 10 & 15 & 5 & 7 \\ 3 & 4 & 6 & b & 3 \\ 4 & 2 & 3 & 1 & a \end{bmatrix} \overset{III_R \leftrightarrow I_R}{\sim}$$

$$\sim \begin{bmatrix} 1 & 4 & 6 & 2 & 1 \\ 5 & 8 & 12 & 4 & 5 \\ 2 & 6 & 9 & 3 & 2 \\ 7 & 10 & 15 & 5 & 7 \\ 3 & 4 & 6 & b & 3 \\ 4 & 2 & 3 & 1 & a \end{bmatrix} \overset{\substack{II_R - I_R \cdot 5 \\ III_R - I_R \cdot 2}}{\sim} \begin{matrix} IV_R - I_R \cdot 7 \\ V_R - I_R \cdot 3 \\ VI_R - I_R \cdot 4 \end{matrix} \begin{bmatrix} 1 & 4 & 6 & 2 & 1 \\ 0 & -12 & -18 & -6 & 0 \\ 0 & -2 & -3 & -1 & 0 \\ 0 & -18 & -27 & -9 & 0 \\ 0 & -8 & -12 & b-6 & 0 \\ 0 & -14 & -21 & -7 & a-4 \end{bmatrix} \overset{II_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -18 & -6 & -12 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -27 & -9 & -18 \\ 0 & 0 & -12 & b-6 & -8 \\ 0 & a-4 & -21 & -7 & -14 \end{bmatrix} \sim$$

$$\overset{II_R \leftrightarrow III_R}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -18 & -6 & -12 \\ 0 & 0 & -27 & -9 & -18 \\ 0 & 0 & -12 & b-6 & -8 \\ 0 & a-4 & -21 & -7 & -14 \end{bmatrix} \overset{III_R - II_R \cdot 6}{\sim} \begin{matrix} IV_R - II_R \cdot 9 \\ V_R - II_R \cdot 4 \\ VI_R - II_R \cdot 7 \end{matrix} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-2 & 0 \\ 0 & a-4 & 0 & 0 & 0 \end{bmatrix}$$

Diskusija

1° $a=4$, $b=2$ $\quad \text{rang} A = 2$

2° $a=4$, $b \neq 2$ $\quad \text{rang} A = 3$

3° $a \neq 4$, $b=2$ $\quad \text{rang} A = 3$

4° $a \neq 4$, $b \neq 2$. $\quad \text{rang} A = 4$

Diskutovati rang matrice

$$M = \begin{bmatrix} 14 & 4 & 2\lambda-4 & -6 \\ 6 & 2 & -1 & -3 \\ 3\lambda+4 & 2 & -2\lambda+1 & -3 \\ 24 & 8 & -4 & -12 \end{bmatrix}$$

za razne vrijednosti parametra λ.

Rj.

$$M = \begin{bmatrix} 14 & 4 & 2\lambda-4 & -6 \\ 6 & 2 & -1 & -3 \\ 3\lambda+4 & 2 & -2\lambda+1 & -3 \\ 24 & 8 & -4 & -12 \end{bmatrix} \overset{"III_v + I_v"}{\sim} \begin{bmatrix} 14 & 4 & 2\lambda-4 & -6 \\ 6 & 2 & -1 & -3 \\ 3\lambda+18 & 6 & -3 & -9 \\ 24 & 8 & -4 & -12 \end{bmatrix} \overset{IV_v : 4}{\sim} \begin{matrix} I_v : 2 \\ III_v : 3 \end{matrix}$$

$$\begin{bmatrix} 7 & 2 & \lambda-2 & -3 \\ 6 & 2 & -1 & -3 \\ \lambda+6 & 2 & -1 & -3 \\ 6 & 2 & -1 & -3 \end{bmatrix} \overset{IV_v - II_v}{\sim} \begin{bmatrix} 7 & 2 & \lambda-2 & -3 \\ 6 & 2 & -1 & -3 \\ \lambda+6 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{I_v \leftrightarrow II_v}{\sim}$$

$$\begin{bmatrix} 6 & 2 & -1 & -3 \\ 7 & 2 & \lambda-2 & -3 \\ \lambda+6 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{I_k \leftrightarrow IV_k}{\sim} \begin{bmatrix} -3 & 2 & -1 & 6 \\ -3 & 2 & \lambda-2 & 7 \\ -3 & 2 & -1 & \lambda+6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{II_v - I_v}{\sim} III_v - I_v$$

$$\sim \begin{bmatrix} -3 & 2 & -1 & 6 \\ 0 & 0 & \lambda-1 & 1 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Za $\lambda = 0$

$\text{rang}(M) = 2$

Za $\lambda \neq 0$ $\quad \text{rang}(M) = 3$

\# Diskutovati rang matrice $\begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix}$ za razne vrijednosti parametra t.

Rj.

$$M = \begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix} \overset{III_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 & t \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 0 & 4 & -3 & 0 \end{bmatrix} \overset{I_v \leftrightarrow IV_v}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 2 & -1 & 0 & t \end{bmatrix} \underset{IV_v - I_v}{\overset{II_v - I_v \cdot 2}{\sim}} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix} \overset{II_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix}$$

$$\overset{IV_v + II_v \cdot 2}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & -9 & 11 & t \end{bmatrix} \overset{IV_v - III_v \cdot \frac{3}{2}}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & t \end{bmatrix}$$

$$-9 + 6 \cdot \frac{3}{2} = -9 + 9 = 0$$

$$11 - 7 \cdot \frac{3}{2} = \frac{22}{2} - \frac{21}{2} = \frac{1}{2}$$

Bez obzira na vrijednost parametra t rang matrice M je uvijek 4.

\# Diskutovati rang matrice $M = \begin{bmatrix} 7-a & -12 & 6 \\ 10 & -19-a & 10 \\ 12 & -24 & 13-a \end{bmatrix}$ u zavisnosti od parametra a.

Rj.

$$M = \begin{bmatrix} 7-a & -12 & 6 \\ 10 & -19-a & 10 \\ 12 & -24 & 13-a \end{bmatrix} \overset{I_k + (II_k + III_k)}{\sim} \begin{bmatrix} 1-a & -12 & 6 \\ 1-a & -19-a & 10 \\ 1-a & -24 & 13-a \end{bmatrix} \underset{III_v - I_v}{\overset{II_v - I_v}{\sim}}$$

$$\sim \begin{bmatrix} 1-a & -12 & 6 \\ 0 & -7-a & 4 \\ 0 & -12 & 7-a \end{bmatrix} \underset{a \neq -7}{\overset{III_v + II_v \cdot \frac{12}{-7-a}}{\sim}} \begin{bmatrix} 1-a & -12 & 6 \\ 0 & -7-a & 4 \\ 0 & 0 & 7-a+4 \cdot \frac{12}{-7-a} \end{bmatrix}$$

$$7-a + \frac{48}{-7-a} = 7-a + \frac{-48}{7+a} = \frac{(7-a)(7+a)-48}{7+a} = \frac{49-a^2-48}{7+a} = \frac{1-a^2}{7+a}$$

$$= \frac{(1-a)(1+a)}{7+a}$$

$$M = \begin{bmatrix} 1-a & -12 & 6 \\ 0 & -7-a & 4 \\ 0 & 0 & \frac{(1-a)(1+a)}{7+a} \end{bmatrix}$$

Diskusija:

1° $a = 1$ $\quad M = \begin{bmatrix} 0 & -12 & 6 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim$

$$\overset{II_v + I_v \cdot \frac{8}{-12}}{\sim} \begin{bmatrix} 0 & -12 & 6 \\ 0 & 0 & 4+6 \cdot \frac{8}{-12} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -12 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rang}\, M = 1$$

2° $a = -1$ $\quad M = \begin{bmatrix} -2 & -12 & 6 \\ 0 & -6 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rang}\, M = 2$

3° $a = -7$ $\quad M = \begin{bmatrix} 8 & -12 & 6 \\ 0 & 0 & 4 \\ 0 & -12 & 14 \end{bmatrix} \overset{II_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 8 & -12 & 6 \\ 0 & -12 & 14 \\ 0 & 0 & 4 \end{bmatrix}$

$$\text{rang}\, M = 3$$

4° $a \neq 1$ i $a \neq -1$ i $a \neq -7$ $\quad \text{rang}\, M = 3$

Inverzna matrica

Transponovanu matricu matrice A obilježavamo sa A^T. Kofaktor A_{ij}, matrice A, elementa a_{ij} je determinanta pomnožena sa $(-1)^{i+j}$ čiji su elementi svi elementi iz matrice A osim one kolone i one vrste u kojoj se nalazi koeficijent a_{ij}.

Npr.

$$A=\begin{bmatrix}3&7&2\\6&8&9\\1&2&4\end{bmatrix},\quad \underbrace{A_{23}=(-1)^{2+3}\begin{vmatrix}3&7\\1&2\end{vmatrix}}_{\text{kofaktor elementa } a_{23}},\quad \overbrace{A_{12}=(-1)^{1+2}\begin{vmatrix}3&2\\1&4\end{vmatrix}}^{\text{kofaktor elementa } a_{12}},\quad \underbrace{A_{31}=(-1)^{3+1}\cdot\begin{vmatrix}7&2\\8&9\end{vmatrix}}_{\text{kofaktor elementa } a_{31}}$$

$$A^T=\begin{bmatrix}3&6&1\\7&8&2\\2&9&4\end{bmatrix}$$

Kofaktor matrica (A_{kof}) kvadratne matrice A je matrica kofaktora A_{ik} elemenata a_{ik} dane matrice.

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\qquad A_{kof}=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}$$

Za matricu A kažemo da je regularna ako je $\det A\neq 0$. Inverznu matricu računamo po formuli:

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

Neke osobine inverzne matrice:

$$A^{-1}\cdot A=A\cdot A^{-1}=I$$

$$(AB)^{-1}=B^{-1}A^{-1}$$

1. Nađi inverznu matricu matrice $A=\begin{bmatrix}1&0&2\\1&1&0\\1&1&1\end{bmatrix}$.

Rj. $A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$

$$\det A=\begin{vmatrix}1&0&2\\1&1&0\\1&1&1\end{vmatrix}\overset{III_R-II_R}{=\!=\!=}\begin{vmatrix}1&0&2\\1&1&0\\0&0&1\end{vmatrix}=1\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}=1$$

$A_{11}=(-1)^2\begin{vmatrix}1&0\\1&1\end{vmatrix}=1$ $\quad A_{13}=(-1)^4\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$ $\quad A_{22}=(-1)^4\begin{vmatrix}1&2\\1&1\end{vmatrix}=-1$

$A_{12}=(-1)^3\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1$ $\quad A_{21}=(-1)^3\begin{vmatrix}0&2\\1&1\end{vmatrix}=2$ $\quad A_{23}=(-1)^5\begin{vmatrix}1&0\\1&1\end{vmatrix}=-1$

$A_{31}=(-1)^4\begin{vmatrix}0&2\\1&0\end{vmatrix}=-2$

$A_{32}=(-1)^5\begin{vmatrix}1&2\\1&0\end{vmatrix}=2$

$A_{33}=(-1)^6\begin{vmatrix}1&0\\1&1\end{vmatrix}=1$

$$A_{kof}=\begin{bmatrix}1&-1&0\\2&-1&-1\\-2&2&1\end{bmatrix}$$

$$\underbrace{A^{-1}=\begin{bmatrix}1&2&-2\\-1&-1&2\\0&-1&1\end{bmatrix}}_{\text{inverzna matrica matrice } A}$$

provjera:

$$\begin{bmatrix}1&0&2\\1&1&0\\1&1&1\end{bmatrix}\cdot\begin{bmatrix}1&2&-2\\-1&-1&2\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

2. Nađi inverznu matricu matrice $B=\begin{bmatrix}3&2&4\\2&3&4\\1&1&2\end{bmatrix}$.

Rj. $B^{-1}=\frac{1}{\det B}B_{kof}^{T}$, $\det B=\begin{vmatrix}3&2&4\\2&3&4\\1&1&2\end{vmatrix}\;\frac{II_k-I_k}{III_k-I_k\cdot 2}\;\begin{vmatrix}3&-1&-2\\2&1&0\\1&0&0\end{vmatrix}=\begin{vmatrix}-1&-2\\1&0\end{vmatrix}=2$

$B_{11}=(-1)^2\begin{vmatrix}3&4\\1&2\end{vmatrix}=2$ $\quad B_{21}=(-1)^2\begin{vmatrix}2&4\\1&2\end{vmatrix}=0$ $\quad B_{31}=(-1)^4\begin{vmatrix}2&4\\3&4\end{vmatrix}=-4$

$B_{12}=(-1)^3\begin{vmatrix}2&4\\1&2\end{vmatrix}=0$ $\quad B_{22}=(-1)^4\begin{vmatrix}3&4\\1&2\end{vmatrix}=2$ $\quad B_{32}=(-1)^5\begin{vmatrix}3&4\\2&4\end{vmatrix}=-4$

$B_{13}=(-1)^4\begin{vmatrix}2&3\\1&1\end{vmatrix}=-1$ $\quad B_{23}=(-1)^5\begin{vmatrix}3&2\\1&1\end{vmatrix}=-1$ $\quad B_{33}=(-1)^6\begin{vmatrix}3&2\\2&3\end{vmatrix}=5$

$$B_{kof}^{T}=\begin{bmatrix}2&0&-4\\0&2&-4\\-1&-1&5\end{bmatrix},\quad B^{-1}=\frac{1}{2}\begin{bmatrix}2&0&-4\\0&2&-4\\-1&-1&5\end{bmatrix}=\begin{bmatrix}1&0&-2\\0&1&-2\\-\frac{1}{2}&-\frac{1}{2}&\frac{5}{2}\end{bmatrix}$$ tražena inverzna matrica

3. Nađi inverznu matricu matrice $C=\begin{bmatrix}2&1\\5&4\end{bmatrix}$.

Rj. $C^{-1}=\frac{1}{\det C}C_{kof}^{T}$, $\det C=\begin{vmatrix}2&1\\5&4\end{vmatrix}=3$

$C_{11}=(-1)^2\cdot 4=4$ $\quad C_{21}=(-1)^3\cdot 1=-1$

$C_{12}=(-1)^3\cdot 5=-5$ $\quad C_{22}=(-1)^4\cdot 2=2$

$$C_{kof}^{T}=\begin{bmatrix}4&-1\\-5&2\end{bmatrix}\qquad C^{-1}=\begin{bmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{5}{3}&\frac{2}{3}\end{bmatrix}$$

4. Nađi inverznu matricu sljedećih matrica:

a) $A=\begin{bmatrix}3&4&4\\1&6&1\\2&3&3\end{bmatrix}$

b) $B=\begin{bmatrix}-3&-1&-1\\1&3&2\\-2&-1&-2\end{bmatrix}$

c) $C=\begin{bmatrix}7&3&3\\6&3&4\\-1&-2&-3\end{bmatrix}$

Rješenja:

a) $A^{-1}=\begin{bmatrix}3&0&-4\\-\frac{1}{5}&\frac{1}{5}&\frac{1}{5}\\-\frac{8}{5}&-\frac{1}{5}&\frac{14}{5}\end{bmatrix}$

b) $B^{-1}=\begin{bmatrix}-\frac{4}{9}&-\frac{1}{9}&\frac{1}{9}\\-\frac{2}{9}&\frac{4}{9}&\frac{5}{9}\\\frac{5}{9}&-\frac{1}{9}&-\frac{8}{9}\end{bmatrix}$

c) $\det C=8$

Pronalaženje inverzne matrice uz pomoć Gauss-Jordan-ovih eliminacija

Posmatrajmo neku proizvoljnu matricu A.

Gauss-Jordan-ove operacije definisane na proizvoljnoj matrici su

(i) množenje proizvoljnog reda matrice brojem različitim od 0

(ii) dodavanje reda i matrice, pomnožen nekim brojem, redu j $(i \neq j)$

Ako je B matrica dobijena iz A pomoću Gauss-Jordanovih operacija pišemo

$$A \xrightarrow{\text{Gauss-Jordan}} B$$

Vrijedi sljedeća teorema

<u>Teorem</u> (računanje inverza)

Za inverznu matricu matrice A vrijedi sljedeća redukcija

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan}} [I \mid A^{-1}] .$$

Ova redukcija neće raditi jedino u slučaju ako se pojavi red nula na lijevoj strani u matrici A, a ovo će se pojaviti ako i samo ako je A singularna matrica. Drugačiji (i nekako mnogo praktičniji) algoritam za pronalaženje inverzne matrice je pomoću LU faktorizacije.

Uz pomoć Gauss-Jordanovih eliminacija izračunati inverznu matricu matrice $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

R:

$$[Q \mid I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{II_v + I_v\cdot(-1)} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{I_v + II_v\cdot(-1)}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Prema tome $Q^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Uz pomoć Gauss-Jordanovih eliminacija izračunati inverznu matricu matrice $Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

R:

$$[Q \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{I_v + II_v\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{II_v + III_v\cdot(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

\# Uz pomoć Gauss-Jordanovih eliminacija izračunati inverznu matricu matrice $P=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}$.

Rj.:

$$[P \mid I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&2&2&0&1&0\\1&2&3&0&0&1\end{array}\right]\xrightarrow[III_v+I_v\cdot(-1)]{II_v+I_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&1&2&-1&0&1\end{array}\right]$$

$$\xrightarrow{III_v+II_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&0&1&0&-1&1\end{array}\right]\xrightarrow{I_v+II_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&0&0&2&-1&0\\0&1&1&-1&1&0\\0&0&1&0&-1&1\end{array}\right]$$

$$\xrightarrow{II_v+III_v\cdot(-1)}\left[\begin{array}{ccc|ccc}1&0&0&2&-1&0\\0&1&0&-1&2&-1\\0&0&1&0&-1&1\end{array}\right]$$

Prema tome $P^{-1}=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$.

Matrične jednačine

U sljedećim primjerima neka su A, B, C, X neke date kvadratne matrice.

$A^{-1}\cdot B \neq B\cdot A^{-1}$

$A\cdot B \neq B\cdot A$

Da bismo odredili nepoznatu X u matričnoj jednačini prvo trebamo izvesti formulu za nepoznatu X.

Matrice se ne mogu dijeliti.

\# $A\cdot X = B \quad /\cdot A^{-1}$ sa lijeve strane

$A^{-1}\cdot A\cdot X = A^{-1}\cdot B$

$I\cdot X = A^{-1}\cdot B$

$X = A^{-1}\cdot B$

\# $A\cdot X\cdot B = C \quad /\cdot A^{-1}$ sa lijeve strane

$A^{-1}\cdot A\cdot X\cdot B = A^{-1}\cdot C$

$I\cdot X\cdot B = A^{-1}\cdot C \quad /B^{-1}$ sa desne strane

$X\cdot B\cdot B^{-1} = A^{-1}\cdot C\cdot B^{-1}$

$X\cdot I = A^{-1}\cdot C\cdot B^{-1}$

$X = A^{-1}\cdot C\cdot B^{-1}$

\# $A\cdot X + I = X - 2I$

$AX - X = -I - 2I$

$\underbrace{(A-I)}_{B}X = -3I$

$BX = -3I \quad /\cdot B^{-1}$ sa desne strane

$B^{-1}\cdot B\cdot X = B^{-1}\cdot(-3I)$

$I\cdot X = -3B^{-1}$

$X = -3(A-I)^{-1}$

\# $X^{-1}\cdot A = B^{-1} \quad /\cdot A^{-1}$ sa desne strane

$X^{-1}\cdot A\cdot A^{-1} = B^{-1}\cdot A^{-1}$

$X^{-1}\cdot I = B^{-1}\cdot A^{-1}$

$X^{-1} = B^{-1}\cdot A^{-1} \quad /^{(-1)}$

$X = A\cdot B$

\# $A^{-1}\cdot X = X - I$

$A^{-1}\cdot X - X = -I$

$\underbrace{(A^{-1}-I)}_{B}\cdot X = -I$

$BX = -I \quad /\cdot B^{-1}$ sa lijeve strane

$B^{-1}\cdot B\cdot X = -B^{-1}\cdot I$

$X = -B^{-1}$

$X = -(A^{-1}-I)^{-1}$

\# $\underbrace{(A+3I)}_{C}(X-I)=B$

$C(X-I)=B$ $/\cdot C^{-1}$ sa lijeve strane

$C^{-1}C(X-I)=C^{-1}\cdot B$

$X-I=C^{-1}\cdot B$

$X=C^{-1}\cdot B+I$

$X=(A+3I)^{-1}\cdot B+I$

\# $B^{-1}\cdot X\cdot A=(3B+2I)^{-1}$

$/\cdot B$ sa lijeve strane

$B\cdot B^{-1}\cdot X\cdot A=B(3B+2I)^{-1}$

$X\cdot A=B(3B+2I)^{-1}$ $/\cdot A^{-1}$ sa desne strane

$X=B(3B+2I)^{-1}\cdot A^{-1}$

\# $(AXB)^{-1}=B^{-1}(X^{-1}+B)$ $/\cdot(AXB)$ sa lijeve strane

$(AXB)(AXB)^{-1}=AX\underbrace{BB^{-1}}_{I}(X^{-1}+B)$

$I=AX(X^{-1}+B)$

$I=AXX^{-1}+AXB$

$I=A+AXB$

$AXB=I-A$ $/\cdot A^{-1}$ sa lijeve str. $/\cdot B^{-1}$ sa desne str.

$A^{-1}AXB\cdot B^{-1}=A^{-1}(I-A)\cdot B^{-1}$

$X=A^{-1}(I-A)\cdot B^{-1}$

1. Riješiti matričnu jednačinu $X\cdot\begin{bmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}$.

Rj. $X\cdot A=B$ $/\cdot A^{-1}$ sa desne str.

$X=B\cdot A^{-1}$, $A=\begin{bmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{bmatrix}$

$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$

$A_{11}=(-1)^2\begin{vmatrix}-2&-3\\3&5\end{vmatrix}=-1$

$A_{12}=(-1)^3\begin{vmatrix}-1&-3\\2&5\end{vmatrix}=-1$

$A_{13}=(-1)^4\begin{vmatrix}-1&-2\\2&3\end{vmatrix}=1$

$A_{21}=(-1)^3\begin{vmatrix}1&1\\3&5\end{vmatrix}=-2$

$A_{22}=(-1)^4\begin{vmatrix}1&1\\2&5\end{vmatrix}=3$

$A_{23}=(-1)^5\begin{vmatrix}1&1\\2&3\end{vmatrix}=-1$

$\det A=\begin{vmatrix}1&1&1\\-1&-2&-3\\2&3&5\end{vmatrix}\overset{I_k-III_k}{\underset{II_k-III_k}{=}}\begin{vmatrix}0&0&1\\2&1&-3\\-3&-2&5\end{vmatrix}=\begin{vmatrix}2&1\\-3&-2\end{vmatrix}=-1.$

$A_{31}=(-1)^4\begin{vmatrix}1&1\\-2&-3\end{vmatrix}=-1$

$A_{32}=(-1)^5\begin{vmatrix}1&1\\-1&-3\end{vmatrix}=2$

$A_{33}=(-1)^6\begin{vmatrix}1&1\\-1&-2\end{vmatrix}=-1$

$A_{kof}^{T}=\begin{bmatrix}-1&-2&-1\\-1&3&2\\1&-1&-1\end{bmatrix}$ $A^{-1}=\begin{bmatrix}1&2&1\\1&-3&-2\\-1&1&1\end{bmatrix}$

$X=B\cdot A^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&2&1\\1&-3&-2\\-1&1&1\end{bmatrix}=\begin{bmatrix}1&-3&-2\\1&2&1\\1&-1&-1\end{bmatrix}$ rješenje matrične jednačine

2. Riješiti matričnu jednačinu $A\cdot X=X+I$ ako je $A=\begin{bmatrix}1&-1&1\\1&3&-2\\3&1&-1\end{bmatrix}$.

Rj. $AX=X+I$

$AX-X=I$

$(A-I)X=I$ $/\cdot(A-I)^{-1}$ sa lijeve strane

$(A-I)(A-I)^{-1}X=(A-I)^{-1}\cdot I$

$X=(A-I)^{-1}$

$C=A-I=\begin{bmatrix}1&-1&1\\1&3&-2\\3&1&-1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&-1&1\\1&2&-2\\3&1&-2\end{bmatrix}$

$C^{-1}=\frac{1}{\det C}C_{kof}^{T}$

$\det C=\begin{vmatrix}0&-1&1\\1&2&-2\\3&1&-2\end{vmatrix}\overset{II_k+III_k}{=}\begin{vmatrix}0&0&1\\1&0&-2\\3&-1&-2\end{vmatrix}=\begin{vmatrix}0&1\\1&-2\end{vmatrix}=-1$

$C_{11}=(-1)^2\begin{vmatrix}2&-2\\1&-2\end{vmatrix}=-2$

$C_{12}=(-1)^3\begin{vmatrix}1&-2\\3&-2\end{vmatrix}=-4$

$C_{13}=(-1)^4\begin{vmatrix}1&2\\3&1\end{vmatrix}=-5$

$C_{21}=(-1)^3\begin{vmatrix}-1&1\\1&-2\end{vmatrix}=-1$

$C_{22}=(-1)^4\begin{vmatrix}0&1\\3&-2\end{vmatrix}=-3$

$C_{23}=(-1)^5\begin{vmatrix}0&-1\\3&1\end{vmatrix}=-3$

$C_{31}=(-1)^4\begin{vmatrix}-1&1\\2&-2\end{vmatrix}=0$

$C_{32}=(-1)^5\begin{vmatrix}0&1\\1&-2\end{vmatrix}=1$

$C_{33}=(-1)^6\begin{vmatrix}0&-1\\1&2\end{vmatrix}=1$

$C_{kof}^{T}=\begin{bmatrix}-2&-1&0\\-4&-3&1\\-5&-3&1\end{bmatrix}$, $C^{-1}=\begin{bmatrix}2&1&0\\4&3&-1\\5&3&-1\end{bmatrix}$, $X=\begin{bmatrix}2&1&0\\4&3&-1\\5&3&-1\end{bmatrix}$ rješenje

3. Riješiti matričnu jednačinu $(A+B)^{-1}AX^{-1}=B^{-1}$, gdje su matrice $A=\begin{bmatrix}1&-1\\2&1\end{bmatrix}$ i $B=\begin{bmatrix}3&0\\-1&2\end{bmatrix}$.

Rj. $(A+B)^{-1}A\cdot X^{-1}=B^{-1}$ $/\cdot(A+B)$ sa lijeve strane

$(A+B)(A+B)^{-1}\cdot A\cdot X^{-1}=(A+B)\cdot B^{-1}$

$A\cdot X^{-1}=(A+B)\cdot B^{-1}$ $/\cdot A^{-1}$ sa lijeve strane

$A^{-1}\cdot A\cdot X^{-1}=A^{-1}(A+B)\cdot B^{-1}$

$X^{-1}=A^{-1}(A+B)\cdot B^{-1}$ $/^{-1}$

$X=B(A+B)^{-1}A$

$C=A+B=\begin{bmatrix}4&-1\\1&3\end{bmatrix}$

$$C=\begin{bmatrix}4 & -1\\ 1 & 3\end{bmatrix},\quad C^{-1}=\frac{1}{\det C}C_{kof}^{T},\quad \det C=\begin{vmatrix}4 & -1\\ 1 & 3\end{vmatrix}=13,$$

$$c_{11}=(-1)^2\cdot 3=3$$
$$c_{12}=(-1)^3\cdot 1=-1$$
$$c_{21}=(-1)^3\cdot(-1)=1$$
$$c_{22}=(-1)^4\cdot 4=4$$

$$C_{kof}^{T}=\begin{bmatrix}3 & 1\\ -1 & 4\end{bmatrix},\quad C^{-1}=\begin{bmatrix}\frac{3}{13} & \frac{1}{13}\\ -\frac{1}{13} & \frac{4}{13}\end{bmatrix}$$

$$X=B\cdot C^{-1}\cdot A=\begin{bmatrix}3 & 0\\ -1 & 2\end{bmatrix}\cdot\frac{1}{13}\cdot\begin{bmatrix}3 & 1\\ -1 & 4\end{bmatrix}\cdot\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}3 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}5 & -2\\ 7 & 5\end{bmatrix}=\frac{1}{13}\begin{bmatrix}15 & -6\\ 9 & 12\end{bmatrix}$$

$$X=\begin{bmatrix}\frac{15}{13} & -\frac{6}{13}\\ \frac{9}{13} & \frac{12}{13}\end{bmatrix}$$ rješenje matrične jednačine

4. Riješiti matričnu jednačinu $(A+3I)(X-I)=B$, ako je $A=\begin{bmatrix}-2 & 5 & -2\\ 2 & 8 & 0\\ -1 & -5 & -2\end{bmatrix}$, $B=\begin{bmatrix}-3 & 21 & 1\\ 2 & 50 & -2\\ 1 & -22 & 0\end{bmatrix}$ i I jedinična matrica.

Rj. $(A+3I)(X-I)=B \quad /\cdot(A+3I)^{-1}$ sa lijeve strane

$$(A+3I)^{-1}(A+3I)(X-I)=(A+3I)^{-1}\cdot B$$
$$X-I=(A+3I)^{-1}\cdot B$$
$$X=(A+3I)^{-1}\cdot B+I$$

$$C^{-1}=\frac{1}{\det C}C_{kof}^{T}$$

$$\det C=\begin{vmatrix}1 & 5 & -2\\ 2 & 11 & 0\\ -1 & -5 & 1\end{vmatrix}\ \underline{\underline{I_k+III_k}}$$
$$=\begin{vmatrix}0 & 0 & -1\\ 2 & 11 & 0\\ -1 & -5 & 1\end{vmatrix}=(-1)\begin{vmatrix}2 & 11\\ -1 & -5\end{vmatrix}=-1$$

$$C=A+3I=\begin{bmatrix}-2 & 5 & -2\\ 2 & 8 & 0\\ -1 & -5 & -2\end{bmatrix}+\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}=\begin{bmatrix}1 & 5 & -2\\ 2 & 11 & 0\\ -1 & -5 & 1\end{bmatrix}$$

$$c_{11}=(-1)^2\begin{vmatrix}11 & 0\\ -5 & 1\end{vmatrix}=11\qquad c_{12}=(-1)^3\begin{vmatrix}2 & 0\\ -1 & 1\end{vmatrix}=-2\qquad c_{13}=(-1)^4\begin{vmatrix}2 & 11\\ -1 & -5\end{vmatrix}=1$$

$$c_{21}=(-1)^3\begin{vmatrix}5 & -2\\ -5 & 1\end{vmatrix}=5\qquad c_{31}=(-1)^4\begin{vmatrix}5 & -2\\ 11 & 0\end{vmatrix}=22$$

$$c_{22}=(-1)^4\begin{vmatrix}1 & -2\\ -1 & 1\end{vmatrix}=-1\qquad c_{32}=(-1)^5\begin{vmatrix}1 & -2\\ 2 & 0\end{vmatrix}=-4$$

$$c_{23}=(-1)^5\begin{vmatrix}1 & 5\\ -1 & -5\end{vmatrix}=0\qquad c_{33}=(-1)^6\begin{vmatrix}1 & 5\\ 2 & 11\end{vmatrix}=1\qquad C_{kof}^{T}=\begin{bmatrix}11 & 5 & 22\\ -2 & -1 & -4\\ 1 & 0 & 1\end{bmatrix}$$

$$C^{-1}=\begin{bmatrix}-11 & -5 & -22\\ 2 & 1 & 4\\ -1 & 0 & -1\end{bmatrix}\qquad C^{-1}\cdot B=\begin{bmatrix}-11 & -5 & -22\\ 2 & 1 & 4\\ -1 & 0 & -1\end{bmatrix}\cdot\begin{bmatrix}-3 & 21 & 1\\ 2 & 50 & -2\\ 1 & -22 & 0\end{bmatrix}=\begin{bmatrix}1 & 3 & -1\\ 0 & 4 & 0\\ 2 & 1 & -1\end{bmatrix}$$

$$X=(A+3I)^{-1}\cdot B+I=\begin{bmatrix}1 & 3 & -1\\ 0 & 4 & 0\\ 2 & 1 & -1\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 3 & -1\\ 0 & 5 & 0\\ 2 & 1 & 0\end{bmatrix}$$ rješenje matrične jednačine

5. Riješiti matričnu jednačinu $(X^{-1}+B^{-1})^{-1}=AX$ ako su $A=\begin{bmatrix}3 & 3 & 2\\ -4 & 1 & -4\\ -3 & 1 & -3\end{bmatrix}$ i $B=\begin{bmatrix}1 & 0 & 2\\ 1 & -2 & 0\\ 0 & -1 & 1\end{bmatrix}$.

Rj. $(X^{-1}+B^{-1})^{-1}=AX \quad /(X^{-1}+B^{-1})$ sa desne strane

$$(X^{-1}+B^{-1})^{-1}\cdot(X^{-1}+B^{-1})=AX(X^{-1}+B^{-1})$$
$$I=A+AXB^{-1}$$
$$AXB^{-1}=I-A \quad /\cdot A^{-1} \text{ sa lijeve str.},\ \cdot B \text{ sa desne str.}$$
$$A^{-1}\cdot A\cdot X\cdot B^{-1}\cdot B=A^{-1}(I-A)\cdot B$$
$$X=A^{-1}(I-A)\cdot B$$

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{-1}$$

$$\det A=\begin{vmatrix}3 & 3 & 2\\ -4 & 1 & -4\\ -3 & 1 & -3\end{vmatrix}\ \underline{\underline{I_k-III_k}}$$
$$=\begin{vmatrix}1 & 3 & 2\\ 0 & 1 & -4\\ 0 & 1 & -3\end{vmatrix}=\begin{vmatrix}1 & -4\\ 1 & -3\end{vmatrix}$$
$$=-3+4=1$$

$$A_{11}=(-1)^2\begin{vmatrix}1 & -4\\ 1 & -3\end{vmatrix}=1$$
$$A_{12}=(-1)^3\begin{vmatrix}-4 & -4\\ -3 & -3\end{vmatrix}=0$$
$$A_{13}=(-1)^4\begin{vmatrix}-4 & 1\\ -3 & 1\end{vmatrix}=-1$$

$$A_{21}=(-1)^3\begin{vmatrix}3 & 2\\ 1 & -3\end{vmatrix}=11\qquad A_{31}=(-1)^4\begin{vmatrix}3 & 2\\ 1 & -4\end{vmatrix}=-14$$

$$A_{22}=(-1)^4\begin{vmatrix}3 & 2\\ -3 & -3\end{vmatrix}=-3\qquad A_{32}=(-1)^5\begin{vmatrix}3 & 2\\ -4 & -4\end{vmatrix}=4$$

$$A_{23}=(-1)^5\begin{vmatrix}3 & 3\\ -3 & 1\end{vmatrix}=-12\qquad A_{33}=(-1)^6\begin{vmatrix}3 & 3\\ -4 & 1\end{vmatrix}=15\qquad A_{kof}=\begin{bmatrix}1 & 0 & -1\\ 11 & -3 & -12\\ -14 & 4 & 15\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1 & 11 & -14\\ 0 & -3 & 4\\ -1 & -12 & 15\end{bmatrix},\quad X=A^{-1}(I-A)\cdot B=\begin{bmatrix}1 & 11 & -14\\ 0 & -3 & 4\\ -1 & -12 & 15\end{bmatrix}\begin{bmatrix}-2 & -3 & -2\\ 4 & 0 & 4\\ 3 & -1 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 1 & -2 & 0\\ 0 & -1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 11 & -14\\ 0 & -3 & 4\\ -1 & -12 & 15\end{bmatrix}\begin{bmatrix}-5 & 8 & -6\\ 4 & -4 & 12\\ 2 & -2 & 10\end{bmatrix}=\begin{bmatrix}11 & -8 & -14\\ -4 & 4 & 4\\ -13 & 10 & 12\end{bmatrix}$$ rješenje matrične jednačine

6. Riješiti matričnu jednačinu:

$$X\cdot\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ 4 & 3 & 2\end{bmatrix}+\begin{bmatrix}1 & 4 & 5\\ 0 & 1 & 4\\ 3 & 4 & 2\end{bmatrix}=X\cdot\begin{bmatrix}1 & 4 & 5\\ 0 & 1 & 4\\ 3 & 4 & 2\end{bmatrix}.$$

Ako označimo $A=\begin{bmatrix}2&3&4\\1&2&3\\4&3&2\end{bmatrix}$ i $B=\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}$ imamo

$$XA+B=XB$$
$$XA-XB=-B$$
$$X(A-B)=-B \quad /\cdot(A-B)^{-1} \text{ sa desne strane}$$
$$X=-B(A-B)^{-1}$$

$$C=A-B=\begin{bmatrix}1&-1&-1\\1&1&-1\\1&-1&0\end{bmatrix}$$

$$\det C=\begin{vmatrix}1&-1&-1\\1&1&-1\\1&-1&0\end{vmatrix}\overset{I_k+II_k}{=\!=}\begin{vmatrix}0&-1&-1\\2&1&-1\\0&-1&0\end{vmatrix}=(-2)\begin{vmatrix}-1&-1\\-1&0\end{vmatrix}=(-2)\cdot(-1)=2$$

$C_{11}=(-1)^2\begin{vmatrix}1&-1\\-1&0\end{vmatrix}=-1$ $\quad C_{21}=1$ $\quad C_{31}=2$

$C_{12}=(-1)^3\cdot 1=-1$ $\quad C_{22}=1$ $\quad C_{32}=0$

$C_{13}=-2$ $\quad C_{23}=0$ $\quad C_{33}=2$

$$C^{-1}=\frac{1}{\det C}\cdot C_{kof}^{T}$$

$$C^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&2\\-1&1&0\\-2&0&2\end{bmatrix},\qquad X=-B\cdot C^{-1}=-\begin{bmatrix}1&4&5\\0&1&4\\3&4&2\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}-1&1&2\\-1&1&0\\-2&0&2\end{bmatrix}$$

$$X=-\frac{1}{2}\begin{bmatrix}-15&5&12\\-9&1&8\\-11&7&10\end{bmatrix}=\begin{bmatrix}\frac{15}{2}&-\frac{5}{2}&-6\\\frac{9}{2}&-\frac{1}{2}&-4\\\frac{11}{2}&-\frac{7}{2}&-5\end{bmatrix}$$ rješenje matrične jednačine

7. Riješiti matričnu jednačinu $(A+X)(B-2I)=A$, ako su $A=\begin{bmatrix}-2&-3&-4\\1&-2&3\\4&3&-2\end{bmatrix}$, $B=\begin{bmatrix}3&-2&1\\1&-1&0\\1&-2&2\end{bmatrix}$, I jedinična matrica.

8. Riješiti matričnu jednačinu $A^{-1}X+B=AX$, ako su $A=\begin{bmatrix}-2&5\\-1&3\end{bmatrix}$; $B=\begin{bmatrix}4&5\\6&7\end{bmatrix}$.

9. Riješiti matričnu jednačinu $(XB^{-1})^{-1}=X^{-1}+A$, ako su $A=\begin{bmatrix}-1&3&1\\1&2&3\\0&3&2\end{bmatrix}$; $B=\begin{bmatrix}1&1&1\\5&5&4\\3&2&1\end{bmatrix}$.

Rješenja:

7. $X=\begin{bmatrix}-2&10&-1\\2&2&-5\\-6&-14&19\end{bmatrix}$ $\quad$ 8. $X=\begin{bmatrix}4&5\\6&7\end{bmatrix}$ $\quad$ 9. $X=\begin{bmatrix}3&-\frac{3}{2}&-\frac{17}{2}\\1&-1&-5\\0&\frac{5}{2}&\frac{15}{2}\end{bmatrix}$

#) Data je matrična jednačina $A(X-B)^{-1}=B^{-1}A$ i matrice $A=\begin{bmatrix}1&1&1\\0&1&1\\0&-1&1\end{bmatrix}$; $B=\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}$.

a) Koji uslov moraju zadovoljavat matrice A i B da bi data jednačina imala rješenje $X=2B$?

b) Riješiti datu jednačinu ako matrice A i B ne zadovoljavaju uslov dobijen pod a)

Rj. a) $A(X-B)^{-1}=B^{-1}A$

$X=2B$

$A\cdot B^{-1}=B^{-1}A$ uslov koji moraju zadovoljavati matrice A i B da bi data matrica imala rješenje $X=2B$.

Uslov možemo pisati i na drugi način:

$A=B^{-1}AB$

ili

$B=A^{-1}\cdot B\cdot A$

b) $A(X-B)^{-1}=B^{-1}A \quad /\cdot(X-B)$ sa desne str

$B^{-1}A(X-B)=A \quad /\cdot B$ sa lijeve str.

$A(X-B)=BA \quad /\cdot A^{-1}$ sa lijeve str.

$X-B=A^{-1}BA$

$X=A^{-1}BA+B$

i odavdje možemo pročitati uslov koji smo dobili pod a) (ako je $B=A^{-1}BA$ tada jednačina ima rješenje $X=2B$)

Provjerimo da li je $B=A^{-1}BA$.

Nađimo prvo A^{-1}

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

$$\det A=\begin{vmatrix}1&1&1\\0&1&1\\0&-1&1\end{vmatrix}=\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2$$

$A_{11}=(-1)^2\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2$ $\quad A_{21}=(-1)^3\begin{vmatrix}1&1\\-1&1\end{vmatrix}=-2$ $\quad A_{31}=(-1)^4\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$

$A_{12}=(-1)^3\begin{vmatrix}0&1\\0&1\end{vmatrix}=0$ $\quad A_{22}=(-1)^4\begin{vmatrix}1&1\\0&1\end{vmatrix}=1$ $\quad A_{32}=(-1)^5\begin{vmatrix}1&1\\0&1\end{vmatrix}=-1$

$A_{13}=(-1)^4\begin{vmatrix}0&1\\0&-1\end{vmatrix}=0$ $\quad A_{23}=(-1)^5\begin{vmatrix}1&1\\0&-1\end{vmatrix}=1$ $\quad A_{33}=(-1)^6\begin{vmatrix}1&1\\0&1\end{vmatrix}=1$

$$A_{kof}=\begin{bmatrix}2&0&0\\-2&1&1\\0&-1&1\end{bmatrix}\qquad A_{kof}^{T}=\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}$$

$$A^{-1}=2\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix},\qquad B\cdot A=\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&2&2\\2&3&3\\3&0&4\end{bmatrix}$$

$$A^{-1}\cdot B\cdot A=2\begin{bmatrix}2&-2&0\\0&1&-1\\0&1&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&3&3\\3&0&4\end{bmatrix}=2\begin{bmatrix}-2&-2&-2\\-1&3&-1\\5&3&7\end{bmatrix}=\begin{bmatrix}-1&-1&-1\\-\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}\\\frac{5}{2}&\frac{3}{2}&\frac{7}{2}\end{bmatrix}$$

ovdje vidimo da matrice A i B ne zadovoljavaju uslov dobijen pod a)

$$X=A^{-1}BA+B=\begin{bmatrix}-1&-1&-1\\-\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}\\\frac{5}{2}&\frac{3}{2}&\frac{7}{2}\end{bmatrix}+\begin{bmatrix}1&1&0\\2&1&0\\3&-1&2\end{bmatrix}=\begin{bmatrix}0&0&-1\\\frac{3}{2}&\frac{5}{2}&-\frac{1}{2}\\\frac{11}{2}&\frac{1}{2}&\frac{11}{2}\end{bmatrix}$$ rješenje matrične jednačine

Riješiti matričnu jednačinu $X\cdot A^{-1}=B^{-1}$ ako su

$$A=\begin{bmatrix}2&3&4\\3&2&1\\1&1&-4\end{bmatrix} \quad i \quad B=\begin{bmatrix}2&1&-1\\2&1&-2\\-1&0&1\end{bmatrix}.$$

Rj.

$$X\cdot A^{-1}=B^{-1} \quad /\cdot A \text{ sa desne strane}$$

$$X\underbrace{A^{-1}\cdot A}_{I}=B^{-1}\cdot A$$

$$X=B^{-1}\cdot A$$

$$\det B=\begin{vmatrix}2&1&-1\\2&1&-2\\-1&0&1\end{vmatrix} \quad \underline{\underline{I_R-II_R}}$$

$$=\begin{vmatrix}0&0&1\\2&1&-2\\-1&0&1\end{vmatrix}=\begin{vmatrix}2&1\\-1&0\end{vmatrix}=0+1$$

$$\det B=1$$

$$B^{-1}=\frac{1}{\det B}B_{kof}^T$$

$$B_{11}=(-1)^2\begin{vmatrix}1&-2\\0&1\end{vmatrix}=1 \qquad B_{21}=(-1)^3\begin{vmatrix}1&-1\\0&1\end{vmatrix}=-1 \qquad B_{31}=(-1)^4\begin{vmatrix}1&-1\\1&-2\end{vmatrix}=-1$$

$$B_{12}=(-1)^3\begin{vmatrix}2&-2\\-1&1\end{vmatrix}=0 \qquad B_{22}=(-1)^4\begin{vmatrix}2&-1\\-1&1\end{vmatrix}=1 \qquad B_{32}=(-1)^5\begin{vmatrix}2&-1\\2&-2\end{vmatrix}=2$$

$$B_{13}=(-1)^4\begin{vmatrix}2&1\\-1&0\end{vmatrix}=1 \qquad B_{23}=(-1)^5\begin{vmatrix}2&1\\-1&0\end{vmatrix}=-1 \qquad B_{33}=(-1)^6\begin{vmatrix}2&1\\2&1\end{vmatrix}=0$$

$$B_{kof}=\begin{bmatrix}1&0&1\\-1&1&-1\\-1&2&0\end{bmatrix}, \quad B_{kof}^T=\begin{bmatrix}1&-1&-1\\0&1&2\\1&-1&0\end{bmatrix}, \quad B^{-1}=\begin{bmatrix}1&-1&-1\\0&1&2\\1&-1&0\end{bmatrix}$$

$$X=B^{-1}\cdot A=\begin{bmatrix}1&-1&-1\\0&1&2\\1&-1&0\end{bmatrix}\begin{bmatrix}2&3&4\\3&2&1\\1&1&-4\end{bmatrix}=\begin{bmatrix}-2&0&7\\5&4&-7\\-1&1&3\end{bmatrix}$$

2-3-1	0+3+2	2-3+0
3-2-1	0+2+2	3-2+0
4-1+4	0+1-8	4-1+0

$$X=\begin{bmatrix}-2&0&7\\5&4&-7\\-1&1&3\end{bmatrix}$$

traženo rješenje

Riješiti matričnu jednačinu $(A+I)^{-1}\cdot X\cdot(3A+I)=2A$ gdje je I jedinična matrica drugog reda a

$$A=\begin{bmatrix}7&8\\-6&-7\end{bmatrix}.$$

Rj.

$$(A+I)^{-1}\cdot X\cdot(3A+I)=2A \quad /(A+I) \text{ sa lijeve strane}$$

$$X\cdot(3A+I)=(A+I)\cdot 2A \quad /\cdot(3A+I)^{-1} \text{ sa desne strane}$$

$$X=(A+I)\cdot 2A\cdot(3A+I)^{-1}$$

$$A=\begin{bmatrix}7&8\\-6&-7\end{bmatrix} \qquad I=\begin{bmatrix}1&0\\0&1\end{bmatrix} \qquad A+I=\begin{bmatrix}8&8\\-6&-6\end{bmatrix}$$

20·22
40
40
440

$$3A+I=\begin{bmatrix}22&24\\-18&-20\end{bmatrix} \qquad 3A=\begin{bmatrix}21&24\\-18&-21\end{bmatrix}$$

18·24
72
36
432

Označimo sa $B=3A-I$ pa pronađimo B^{-1}

$$B^{-1}=\frac{1}{\det B}\cdot B_{kof}^T \qquad \det B=\begin{vmatrix}22&24\\-18&-20\end{vmatrix}=-440+432=-8$$

$$B_{11}=(-1)^2\cdot(-20)=-20 \qquad B_{21}=(-1)^3\cdot 24=-24$$

$$B_{12}=(-1)^3\cdot(-18)=18 \qquad B_{22}=(-1)^4\cdot 22=22 \qquad B_{kof}=\begin{bmatrix}-20&18\\-24&22\end{bmatrix}$$

$$B^{-1}=\frac{-1}{8}\begin{bmatrix}-20&-24\\18&22\end{bmatrix}=(3A+I)^{-1}$$

$$X=(A+I)\cdot 2A\cdot(3A+I)^{-1}=\begin{bmatrix}8&8\\-6&-6\end{bmatrix}\cdot 2\cdot\begin{bmatrix}7&8\\-6&-7\end{bmatrix}\cdot\frac{-1}{8}\begin{bmatrix}-20&-24\\18&22\end{bmatrix}$$

$$=2\begin{bmatrix}4&4\\-3&-3\end{bmatrix}\cdot 2\begin{bmatrix}7&8\\-6&-7\end{bmatrix}\cdot\frac{-1}{8}\cdot 2\begin{bmatrix}-10&-12\\9&11\end{bmatrix}=8\cdot\frac{-1}{8}\begin{bmatrix}4&4\\-3&-3\end{bmatrix}\begin{bmatrix}7&8\\-6&-7\end{bmatrix}\begin{bmatrix}-10&-12\\9&11\end{bmatrix}$$

$$=(-1)\begin{bmatrix}4&4\\-3&-3\end{bmatrix}\begin{bmatrix}-10&-12\\9&11\end{bmatrix}=(-1)\begin{bmatrix}-4&-4\\3&3\end{bmatrix}=\begin{bmatrix}4&4\\-3&-3\end{bmatrix}$$

rješenje matrične jednačine

(#) Riješiti matričnu jednačinu $(AXB)^{-1} = B^{-1}(X^{-1}+B)$ ako je

$$A=\begin{bmatrix}3&-4&5\\2&-3&1\\3&-5&-1\end{bmatrix},\quad B=\begin{bmatrix}1&2&2\\2&1&-2\\2&-2&1\end{bmatrix}.$$

Rj.

$$(AXB)^{-1} = B^{-1}(X^{-1}+B)$$

$$B^{-1}X^{-1}A^{-1} = B^{-1}X^{-1}+B^{-1}\cdot B \qquad /\cdot B \text{ sa lijeve strane}$$

$$X^{-1}A^{-1} = X^{-1}+B$$

$$X^{-1}A^{-1}-X^{-1} = B$$

$$X^{-1}(A^{-1}-I) = B \qquad /\cdot(A^{-1}-I)^{-1} \text{ sa desne strane}$$

$$X^{-1} = B(A^{-1}-I)^{-1} \qquad /^{-1}$$

$$X = (A^{-1}-I)\cdot B^{-1}$$

$$\det A=\begin{vmatrix}3&-4&5\\2&-3&1\\3&-5&-1\end{vmatrix}\overset{I_k+II_k}{=\!=}\begin{vmatrix}-1&-4&5\\-1&-3&1\\-2&-5&-1\end{vmatrix}\underset{III_v-I_v\cdot 2}{\overset{II_v-I_v}{=\!=}}$$

$$=\begin{vmatrix}-1&-4&5\\0&1&-4\\0&3&-11\end{vmatrix}=(-1)\begin{vmatrix}1&-4\\3&-11\end{vmatrix}=(-1)(-11+12)=-1$$

$$A^{-1}=\frac{1}{\det A}\cdot A_{kof}^{T}$$

$$A_{11}=(-1)^2\begin{vmatrix}-3&1\\-5&-1\end{vmatrix}=3+5=8$$

$$A_{12}=(-1)^3\begin{vmatrix}2&1\\3&-1\end{vmatrix}=-(-2-3)=5$$

$$A_{13}=(-1)^4\begin{vmatrix}2&-3\\3&-5\end{vmatrix}=-10+9=-1$$

$$A_{21}=(-1)^3\begin{vmatrix}-4&5\\-5&-1\end{vmatrix}=-(4+25)=-29$$

$$A_{22}=(-1)^4\begin{vmatrix}3&5\\3&-1\end{vmatrix}=-3-15=-18$$

$$A_{23}=(-1)^5\begin{vmatrix}3&-4\\3&-5\end{vmatrix}=-(-15+12)=3$$

$$A_{31}=11 \qquad A_{32}=7 \qquad A_{33}=-1$$

$$A_{kof}=\begin{bmatrix}8&5&-1\\-29&-18&3\\11&7&-1\end{bmatrix}. \qquad A^{-1}=(-1)\begin{bmatrix}8&-29&11\\5&-18&7\\-1&3&-1\end{bmatrix}=\begin{bmatrix}-8&29&-11\\-5&18&-7\\1&-3&1\end{bmatrix}.$$

$$\det B=\begin{vmatrix}1&2&2\\2&1&-2\\2&-2&1\end{vmatrix}\underset{III_v-I_v\cdot 2}{\overset{II_v-I_v\cdot 2}{=\!=}}\begin{vmatrix}1&2&2\\0&-3&-6\\0&-6&-3\end{vmatrix}=\begin{vmatrix}-3&-6\\-6&-3\end{vmatrix}=9-36=-27$$

$$B^{-1}=\frac{1}{\det B}\cdot B_{kof}^{T}\underset{\text{ZA VJEŽBU (slično)}}{=\!=}\frac{(-1)}{-27}\begin{bmatrix}3&6&6\\6&3&-6\\6&-6&3\end{bmatrix}=\frac{1}{27}\cdot 3\cdot\begin{bmatrix}1&2&2\\2&1&-2\\2&-2&1\end{bmatrix}$$

$$B^{-1}=\frac{1}{9}\begin{bmatrix}1&2&2\\2&1&-2\\2&-2&1\end{bmatrix}$$

$$A^{-1}-I=\begin{bmatrix}-8&29&-11\\-5&18&-7\\1&-3&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-9&29&-11\\-5&17&-7\\1&-3&0\end{bmatrix}$$

$$X=(A^{-1}-I)\cdot B^{-1}=\begin{bmatrix}-9&29&-11\\-5&17&-7\\1&-3&0\end{bmatrix}\cdot\frac{1}{9}\begin{bmatrix}1&2&2\\2&1&-2\\2&-2&1\end{bmatrix}=$$

$$=\frac{1}{9}\begin{bmatrix}27&33&-87\\15&21&-51\\-5&-1&8\end{bmatrix}$$

$$X=\begin{bmatrix}3&\frac{11}{3}&-\frac{29}{3}\\\frac{5}{3}&\frac{7}{3}&-\frac{17}{3}\\-\frac{5}{9}&-\frac{1}{9}&\frac{8}{9}\end{bmatrix}$$ rješenje matrične jednačine

Riješiti matričnu jednačinu $A\cdot X^{-1}\cdot B = B\cdot A$, ako je $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ i $B=\begin{bmatrix}1&0\\1&1\end{bmatrix}$.

Rj.

$AX^{-1}B = B\cdot A$ $\quad /\cdot A^{-1}$ sa lijeve strane

$X^{-1}B = A^{-1}B\cdot A$ $\quad /\cdot B^{-1}$ sa desne strane

$X^{-1} = A^{-1}B\cdot A\cdot B^{-1}$ $\quad /^{-1}$

$X = BA^{-1}B^{-1}A$

$A^{-1} = \frac{1}{\det A}\cdot A_{kof}^{T}$ $\qquad \det A = \begin{vmatrix}1&1\\0&1\end{vmatrix} = 1$ $\qquad A_{kof} = \begin{bmatrix}1&0\\-1&1\end{bmatrix}$

$A_{11}=1 \qquad A_{21}=-1$

$A_{12}=0 \qquad A_{22}=1$

$A_{kof}^{T} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$ $\qquad A^{-1} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$

$B^{-1} = \frac{1}{\det B}\cdot B_{kof}^{T}$ $\qquad B_{11}=1$, $B_{12}=-1$, $B_{21}=0$, $B_{22}=1$

$B_{kof} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$ $\qquad B_{kof}^{T} = \begin{bmatrix}1&0\\-1&1\end{bmatrix}$ $\qquad B^{-1} = \begin{bmatrix}1&0\\-1&1\end{bmatrix}$

$\det B = \begin{vmatrix}1&0\\1&1\end{vmatrix} = 1$

$B\cdot A^{-1} = \begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}1&-1\\0&1\end{bmatrix} = \begin{bmatrix}1&-1\\1&0\end{bmatrix}$

$B^{-1}\cdot A = \begin{bmatrix}1&0\\-1&1\end{bmatrix}\cdot\begin{bmatrix}1&1\\0&1\end{bmatrix} = \begin{bmatrix}1&1\\-1&0\end{bmatrix}$

$X = BA^{-1}\cdot B^{-1}A =$

$= \begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\-1&0\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$

$X = \begin{bmatrix}2&1\\1&1\end{bmatrix}$ traženo rješenje

Riješiti matričnu jednačinu: $AX - 2B = 3X + A$ gdje je $A=\begin{bmatrix}6&1&0\\0&5&2\\0&0&4\end{bmatrix}$, $B=\begin{bmatrix}-1&2&0\\2&3&1\\4&0&3\end{bmatrix}$.

Rj. $AX - 2B = 3X + A$

$AX - 3X = 2B + A$

$\underbrace{(A-3I)}_{M}X = \underbrace{2B+A}_{N}$

$MX = N$ $\quad /\cdot M^{-1}$ sa lijeve str.

$M^{-1}MX = M^{-1}\cdot N$

$X = M^{-1}\cdot N$

$M = A - 3I = \begin{bmatrix}6&1&0\\0&5&2\\0&0&4\end{bmatrix} - \begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix} = \begin{bmatrix}3&1&0\\0&2&2\\0&0&1\end{bmatrix}$

$N = 2B + A = \begin{bmatrix}-2&4&0\\4&6&2\\8&0&6\end{bmatrix} + \begin{bmatrix}6&1&0\\0&5&2\\0&0&4\end{bmatrix} = \begin{bmatrix}4&5&0\\4&11&4\\8&0&10\end{bmatrix}$

$M^{-1} = \frac{1}{\det M}\, M_{kof}^{T}$

$\det M = \begin{vmatrix}3&1&0\\0&2&2\\0&0&1\end{vmatrix} = 3\cdot2\cdot1 = 6$

$M_{11} = (-1)^2\begin{vmatrix}2&2\\0&1\end{vmatrix} = 2 \qquad M_{21} = (-1)^3\begin{vmatrix}1&0\\0&1\end{vmatrix} = -1$

$M_{12} = (-1)^3\begin{vmatrix}0&2\\0&1\end{vmatrix} = 0 \qquad M_{22} = (-1)^4\begin{vmatrix}3&0\\0&1\end{vmatrix} = 3$

$M_{13} = (-1)^4\begin{vmatrix}0&2\\0&0\end{vmatrix} = 0 \qquad M_{23} = (1)^5\begin{vmatrix}3&1\\0&0\end{vmatrix} = 0$

$M_{31} = (-1)^4\begin{vmatrix}1&0\\2&2\end{vmatrix} = 2$

$M_{32} = (-1)^5\begin{vmatrix}3&0\\0&2\end{vmatrix} = -6$

$M_{33} = (-1)^6\begin{vmatrix}3&1\\0&2\end{vmatrix} = 6$

$M_{kof} = \begin{bmatrix}2&0&0\\-1&3&0\\2&-6&6\end{bmatrix}$, $\qquad M_{kof}^{T} = \begin{bmatrix}2&-1&2\\0&3&-6\\0&0&6\end{bmatrix}$

$M^{-1} = \frac{1}{6}\begin{bmatrix}2&-1&2\\0&3&-6\\0&0&6\end{bmatrix}$,

$X = M^{-1}\cdot N = \frac{1}{6}\begin{bmatrix}2&-1&2\\0&3&-6\\0&0&6\end{bmatrix}\begin{bmatrix}4&5&0\\4&11&4\\8&0&10\end{bmatrix} = \frac{1}{6}\begin{bmatrix}20&-1&16\\-36&33&48\\48&0&60\end{bmatrix}$

$8-4+16 \qquad 0+12-48$

$10-11+0 \qquad 0+33+0$

$0-4+20 \qquad 12-60$

$X = \begin{bmatrix}\frac{10}{3}&-\frac{1}{6}&\frac{8}{3}\\-6&\frac{11}{2}&8\\8&0&10\end{bmatrix}$ traženo rješenje

Riješiti matričnu jednačinu $(XA+B)^{-1}(XC+B)=C$,

ako je $A=\begin{bmatrix}-1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}$ i $C=\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}$

Rj. $(XA+B)^{-1}(XC+B)=C \quad /(XA+B)$ sa lijeve strane

$$\underbrace{(XA+B)(XA+B)^{-1}}_{I}(XC+B)=(XA+B)\cdot C$$

$$XC+B=XAC+BC$$

$$XC-XAC=BC-B$$

$$X(C-AC)=BC-B \quad /(C-AC)^{-1} \text{ sa desne strane}$$

$$X=B(C-I)(C-AC)^{-1}$$

$$C-1=\begin{bmatrix}-2 & 1 & 2\\ 0 & -2 & 2\\ 0 & 0 & 1\end{bmatrix}$$

$$B(C-1)=\begin{bmatrix}2 & -1 & 1\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & 1 & 2\\ 0 & -2 & 2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-4 & 4 & 3\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}$$

$$AC=\begin{bmatrix}-1 & 2 & 3\\ 0 & 2 & -1\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\\ 0 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & -3 & 8\\ 0 & -2 & 2\\ 0 & 0 & 6\end{bmatrix}$$

Označimo sa $D=C-AC=\begin{bmatrix}-2 & 4 & -6\\ 0 & 1 & 0\\ 0 & 0 & -4\end{bmatrix}$

Izračunajmo D^{-1}. $\quad D^{-1}=\dfrac{1}{\det D}D_{kof}^{T}$

$D_{11}=(-1)^2\begin{vmatrix}1 & 0\\ 0 & -4\end{vmatrix}=-4$ $\quad D_{21}=(-1)^3\begin{vmatrix}4 & -6\\ 0 & -4\end{vmatrix}=16$ $\quad D_{31}=(-1)^4\begin{vmatrix}4 & -6\\ 1 & 0\end{vmatrix}=6$

$D_{12}=(-1)^3\begin{vmatrix}0 & 0\\ 0 & -4\end{vmatrix}=0$ $\quad D_{22}=(-1)^4\begin{vmatrix}-2 & -6\\ 0 & -4\end{vmatrix}=8$ $\quad D_{32}=(-1)^5\begin{vmatrix}-2 & -6\\ 0 & 0\end{vmatrix}=0$

$D_{13}=(-1)^4\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}=0$ $\quad D_{23}=(-1)^5\begin{vmatrix}-2 & 4\\ 0 & 0\end{vmatrix}=0$ $\quad D_{33}=(-1)^6\begin{vmatrix}-2 & 4\\ 0 & 1\end{vmatrix}=-2$

$$\det D=\begin{vmatrix}-2 & 4 & -6\\ 0 & 1 & 0\\ 0 & 0 & -4\end{vmatrix}=(-4)\begin{vmatrix}-2 & 4\\ 0 & 1\end{vmatrix}=8 \qquad D_{kof}=\begin{bmatrix}-4 & 0 & 0\\ 16 & 8 & 0\\ 6 & 0 & -2\end{bmatrix} \qquad D_{kof}^{T}=\begin{bmatrix}-4 & 16 & 6\\ 0 & 8 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$D^{-1}=\begin{bmatrix}-\frac{1}{2} & 2 & \frac{3}{4}\\ 0 & 1 & 0\\ 0 & 0 & -\frac{1}{4}\end{bmatrix}, \quad X=B(C-1)(C-AC)^{-1}=\frac{1}{8}\begin{bmatrix}-4 & 4 & 3\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & 16 & 6\\ 0 & 8 & 0\\ 0 & 0 & -2\end{bmatrix}$$

$$X=\frac{1}{8}\begin{bmatrix}16 & -32 & -30\\ 0 & 16 & 2\\ 0 & 0 & -2\end{bmatrix}=\begin{bmatrix}2 & -4 & -\frac{15}{4}\\ 0 & 2 & \frac{1}{4}\\ 0 & 0 & -\frac{1}{4}\end{bmatrix}$$ traženo rješenje

Riješiti matričnu jednačinu $XAB=C$, $A=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$,

$B=\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 2\\ -1 & 1 & 1\end{bmatrix}$, $C=[0\ 4\ 4]$.

Rj.

$$XAB=C \quad /\cdot(AB)^{-1} \text{ sa desne strane}$$

$$X(AB)(AB)^{-1}=C\cdot(AB)^{-1}$$

$$X=C\cdot(AB)^{-1}$$

$$AB=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 2\\ -1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 2 & 0\\ 1 & 4 & 1\\ -1 & 2 & 3\end{bmatrix}$$

$$\det(AB)=\begin{vmatrix}0 & 2 & 0\\ 1 & 4 & 1\\ -1 & 2 & 3\end{vmatrix}=(-2)\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}=(-2)(3+1)=-8$$

AB označimo sa M, nađimo M^{-1}

$M_{11}=(-1)^2\begin{vmatrix}4 & 1\\ 2 & 3\end{vmatrix}=10$ $\quad M_{21}=(-1)^3\begin{vmatrix}2 & 0\\ 2 & 3\end{vmatrix}=-6$ $\quad M_{31}=(-1)^4\begin{vmatrix}2 & 0\\ 4 & 1\end{vmatrix}=2$

$M_{12}=(-1)^3\begin{vmatrix}1 & 1\\ -1 & 3\end{vmatrix}=-4$ $\quad M_{22}=(-1)^4\begin{vmatrix}0 & 0\\ -1 & 3\end{vmatrix}=0$ $\quad M_{32}=(-1)^5\begin{vmatrix}0 & 0\\ 1 & 1\end{vmatrix}=0$

$M_{13}=(-1)^4\begin{vmatrix}1 & 4\\ -1 & 2\end{vmatrix}=6$ $\quad M_{23}=(-1)^5\begin{vmatrix}0 & 2\\ -1 & 2\end{vmatrix}=-2$ $\quad M_{33}=(-1)^6\begin{vmatrix}0 & 2\\ 1 & 4\end{vmatrix}=-2$

$$M_{kof}=\begin{bmatrix}10 & -4 & 6\\ -6 & 0 & -2\\ 2 & 0 & -2\end{bmatrix}, \quad M_{kof}^{T}=\begin{bmatrix}10 & -6 & 2\\ -4 & 0 & 0\\ 6 & -2 & -2\end{bmatrix}$$

$$(AB)^{-1}=\frac{1}{\det M}\cdot M_{kof}^{T}=\frac{-1}{8}\begin{bmatrix}10 & -6 & 2\\ -4 & 0 & 0\\ 6 & -2 & -2\end{bmatrix}=\begin{bmatrix}-5/4 & 3/4 & -1/4\\ 1/2 & 0 & 0\\ -3/4 & 1/4 & 1/4\end{bmatrix}$$

$$X=C\cdot(AB^{-1})=[0\ 4\ 4]\cdot\left(-\frac{1}{8}\right)\begin{bmatrix}10 & -6 & 2\\ -4 & 0 & 0\\ 6 & -2 & -2\end{bmatrix}=\left(-\frac{1}{8}\right)[8\ -8\ -8]$$

$X=[-1\ 1\ 1]$ rješenje matrične jednačine

Sistem linearnih jednačina

Sistem od m jednačina sa n nepoznatih zovemo sistem linearnih jednačina

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Sisteme linearnih jednačina možemo rješiti:

a) Gausovom metodom

b) Kramerovom metodom (metoda determinanti)

c) Matričnom metodom

d) Kroneker-Kapelijevom metodom

Kroneker-Kapelijeva metoda

Neka je dat sistem linearnih jednačina $Ax=b$, gdje su

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad b=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Matricu $\bar{A}=[A \mid b]$ zovemo proširena matrica.

Teorema (Kroneker-Kapeli):

Sistem ima jedinstveno rješenje ako i samo ako je $\operatorname{rang} A = \operatorname{rang} \bar{A} = n$ (n broj nepoznatih).

Ako je $\operatorname{rang} A = \operatorname{rang} \bar{A} < n$ tada sistem ima ∞ mnogo rješenja. ($n-\operatorname{rang} A$ nepoznatih uzima se proizvoljno)

Ako je $\operatorname{rang} A < \operatorname{rang} \bar{A}$ tada sistem nema rješenja.

1. Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} 2x + 4y - 5z &= -5 \\ -x - y + z &= 0 \\ 2x + y - z &= 1. \end{aligned}$$

Rj.

$$\bar{A}=[A \mid b] = \left[\begin{array}{ccc|c} 2 & 4 & -5 & -5 \\ -1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{array}\right] \underset{I_v \leftrightarrow II_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 2 & 4 & -5 & -5 \\ 2 & 1 & -1 & 1 \end{array}\right] \underset{\substack{II_v + I_v\cdot 2 \\ III_v + I_v \cdot 2}}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 2 & -3 & -5 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\underset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -3 & -5 \end{array}\right] \underset{III_v + II_v \cdot 2}{\sim} \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$\operatorname{rang} A = \operatorname{rang} \bar{A} = 3$

sistem ima jedinstveno rješenje

$$\begin{aligned} -x - y + z &= 0 \\ -y + z &= 1 \\ -z &= -3 \\ \hline z &= 3 \end{aligned}$$

$$\begin{aligned} -x - y &= -3 \\ -y &= -2 \\ \hline y &= 2 \end{aligned}$$

$$\begin{aligned} -x - 2 &= -3 \\ x &= 1 \end{aligned}$$

Rješenje sistema je uređena trojka $(1,2,3)$.

(2.) Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 3 \\ 2x_1 + x_2 \quad &= 2. \end{aligned}$$

Rj. $\bar{A} = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 2 & 1 & 0 & 2 \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \overset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$

$\overset{III_v - II_v \cdot 2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

rang A = rang $\bar{A} = 2 < 3$
sistem ima ∞ mnogo rješenja
3-2 nepoznatih uzimamo proizvoljno

$x_3 = t$
$-x_2 - 2x_3 = 0$
$x_1 + x_2 + x_3 = 1$

$-x_2 - 2t = 0$
$x_2 = -2t$

$x_1 - 2t + t = 1$
$x_1 = t + 1$

Sistem ima beskonačno mnogo rješenja oblika $(t+1, -2t, t)$ gdje je $t \in \mathbb{R}$.

(3.) Kroneker-Kapelijevom metodom rješiti sistem jednačina

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 6z &= 2 \\ 3x + 6y + 9z &= 5. \end{aligned}$$

Rj. $\bar{A} = [A \mid b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 9 & 5 \end{array}\right] \underset{III_v - I_v \cdot 3}{\overset{II_v - I_v \cdot 2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

rang $A = 1$, rang $\bar{A} = 2$, rang $A <$ rang $\bar{A}$
sistem nema rješenja.

(4.) Kroneker-Kapelijevom metodom diskutovati rješenja sistema za razne vrijednosti parametra λ

$$\begin{aligned} \lambda x + y + z &= 1 \\ x + \lambda y + z &= 2 \\ x + y + \lambda z &= -3. \end{aligned}$$

Rj. za $\lambda \in (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$ sistem ima jedinstveno rješenje $\left(\frac{1}{\lambda - 1}, \frac{2}{\lambda - 1}, \frac{-3}{\lambda - 1}\right)$

za $\lambda = -2$ sistem ima ∞ mnogo rješenja $\left(\frac{3t-4}{3}, \frac{3t-5}{3}, t\right)$, $t \in \mathbb{R}$

za $\lambda = 1$ sistem nema rješenja

(#) Rješiti sistem linearnih jednačina

$$\begin{aligned} x - y + z &= 1 \\ x - y - z &= 2 \\ x + y - z &= 3 \\ x + y + z &= 4 \end{aligned}$$

Rj. - upute:
Rješimo sistem Kroneker-Kapelijevom metodom

$\bar{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\Rightarrow$ rang$(A) <$ rang$(\bar{A})$

Dati sistem nema rješenja

Riješiti sistem linearnih jednačina

$$x - y + z = 2$$
$$x - y - z = 3$$
$$x + y - z = 4$$
$$x + y + z = 5$$

Rj.-upute:

Riješimo sistem Kroneker-Kapelijevom metodom

$$\bar{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 3 \\ 1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 5 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow \operatorname{rang}(A) < \operatorname{rang}(\bar{A})$

Dati sistem nema rješenje.

Riješiti sistem jednačina

$$x_1 + 2x_2 - 4x_3 + 8x_4 + 12x_5 = -10$$
$$3x_1 + 7x_2 - 15x_3 + 30x_4 + 45x_5 = -43$$
$$-2x_1 - 3x_2 + 6x_3 - 12x_4 - 18x_5 = 13$$

Rj.-upute:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\bar{A} = [A \mid b] = \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 8 & 12 & -10 \\ 3 & 7 & -15 & 30 & 45 & -43 \\ -2 & -3 & 6 & -12 & -18 & 13 \end{array}\right] \begin{array}{l} II_v + I_v \cdot (-3) \\ III_v + I_v \cdot 2 \end{array}$$

$$\ldots \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & -3 & 6 \end{array}\right]$$

$\Rightarrow$ $\left.\begin{array}{l} \operatorname{rang}(A) = 3 \\ \operatorname{rang}(\bar{A}) = 3 \\ \text{broj nepoznatih} = 5 \end{array}\right\}$ $\Rightarrow$ sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr. $x_4 = s$, $x_5 = t$, $s, t \in \mathbb{R}$

$x_1 = 4$

$x_2 = 5$

$x_3 = 6 + 2s + 3t$

$x_4 = s$

$x_5 = t$

$s, t \in \mathbb{R}$

Riješiti sistem jednačina

$$\begin{aligned} x_1 + 2x_2 - 4x_3 - 8x_4 - 12x_5 &= -11 \\ -2x_1 - 3x_2 + 5x_3 + 10x_4 + 15x_5 &= 7 \\ -3x_1 - 5x_2 + 10x_3 + 20x_4 + 30x_5 &= 25 \end{aligned}$$

Rj.-upute:

Sistem ćemo riješiti Kroneker-Kapelijevom metodom

$$\overline{A} = [A \mid b] = \left[\begin{array}{ccccc|c} 1 & 2 & -4 & -8 & -12 & -11 \\ -2 & -3 & 5 & 10 & 15 & 7 \\ -3 & -5 & 10 & 20 & 30 & 25 \end{array}\right] \underset{III_v + I_v \cdot 3}{\overset{II_v + I_v \cdot 2}{\sim}}$$

$$\ldots \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 2 & 3 & 7 \end{array}\right]$$

$\Rightarrow$ $\left.\begin{array}{l} \text{rang}(A) = 3 \\ \text{rang}(\overline{A}) = 3 \\ \text{broj nepoznatih} = 5 \end{array}\right\} \Rightarrow$ sistem ima beskonačno mnogo rješenja i dvije promjenjive uzimamo proizvoljno npr. $x_4 = s$, $x_5 = t$

$x_1 = 5$

$x_2 = 6$

$x_3 = 7 - 2s - 3t$

$x_4 = s$

$x_5 = t$

$s, t \in \mathbb{R}$

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ 6x_1 - 3x_2 + x_3 - 4x_4 &= 7 \\ 4x_1 - 2x_2 + 14x_3 - 31x_4 &= \lambda \end{aligned}$$

Rj.

Riješimo sistem Kroneker-Kapelijevom metodom:

$$\overline{C} = [C \mid b] = \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 6 & -3 & 1 & -4 & 7 \\ 4 & -2 & 14 & -31 & \lambda \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 8 & -17 & \lambda - 30 \end{array}\right]$$

$$\overset{III_v + II_v}{\sim} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 0 & 0 & \lambda - 68 \end{array}\right]$$

1° $\lambda - 68 \neq 0$, $\lambda \neq 68$ rang $C = 2$, rang $\overline{C} = 3$; rang $C <$ rang $\overline{C}$. Prema Kroneker-Kapelijevoj teoremi sistem nema rješenje

2° $\lambda - 68 = 0$, $\lambda = 68$ rang C = rang $\overline{C} = 2 < 4$ (broj nepoznatih)

Prema Kroneker-Kapelijevoj teoremi dvije promjenjive uzimamo proizvoljno, npr. $x_4 = t$, $x_1 = s$

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ -8x_3 + 17x_4 &= -38 \end{aligned}$$

$x_4 = t$

$-8x_3 + 17t = -38$

$-8x_3 = -17t - 38$

$x_3 = \frac{17}{8}t + \frac{38}{8} = \frac{17}{8}t + \frac{19}{4}$

$x_1 = s$

$2s - x_2 + 3\left(\frac{17}{8}t + \frac{38}{8}\right) - 7t = 15$

$x_2 = \frac{51t}{8} + \frac{114}{8} + 2s - 7t - 15$

$x_2 = -\frac{5}{8}t - \frac{6}{8} + 2s$

$x_2 = 2s - \frac{5}{8}t - \frac{3}{4}$

Za $\lambda = 68$ rješenje sistema je

$$\left(s,\ 2s - \frac{5}{8}t - \frac{3}{4},\ \frac{17}{8}t + \frac{19}{4},\ t\right),\quad t, s \in \mathbb{R}$$

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in \mathbb{R}$:

$$\begin{aligned}
8x_1+12x_2+7x_3+\lambda x_4&=9\\
6x_1+9x_2+5x_3+6x_4&=7\\
4x_1+6x_2+3x_3+4x_4&=5\\
2x_1+3x_2+2x_3+2x_4&=2.
\end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\overline{B}=[B\,|\,b]=\begin{array}{c}\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}\\ \left[\begin{array}{cccc|c} 8 & 12 & 7 & \lambda & 9\\ 6 & 9 & 5 & 6 & 7\\ 4 & 6 & 3 & 4 & 5\\ 2 & 3 & 2 & 2 & 2\end{array}\right]\end{array} \underset{I_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2\\ 6 & 9 & 5 & 6 & 7\\ 4 & 6 & 3 & 4 & 5\\ 8 & 12 & 7 & \lambda & 9\end{array}\right] \underset{\substack{II_v-I_v\cdot 3\\ III_v-I_v\cdot 2\\ IV_v-I_v\cdot 4}}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2\\ 0 & 0 & -1 & 0 & 1\\ 0 & 0 & -1 & 0 & 1\\ 0 & 0 & -1 & \lambda-8 & 1\end{array}\right] \underset{\substack{III_v-II_v\\ IV_v-II_v}}{\sim} \left[\begin{array}{cccc|c} 2 & 3 & 2 & 2 & 2\\ 0 & 0 & -1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda-8 & 0\end{array}\right]$$

1° za $\lambda=8$ imamo $\operatorname{rang} B=\operatorname{rang}\overline{B}=2<4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Dvije promjenjive uzimamo proizvoljno npr. $x_1=t$, $x_4=s$

$$\begin{aligned}
2x_1+3x_2+2x_3+2x_4&=2\\
-x_3+0x_4&=1
\end{aligned}$$

$x_3=-1$

$2t+3x_2-2+2s=2$

$3x_2=4-2t-2s$

$x_2=\frac{2}{3}(2-t-s)$

Rješenje sistema (za $\lambda=8$) je $(t, \frac{2}{3}(2-t-s), -1, s)$ gdje su $s,t\in\mathbb{R}$.

2° za $\lambda\neq 8$ imamo $\operatorname{rang} B=\operatorname{rang}\overline{B}=3<4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. Jednu promjenjivu uzimamo proizvoljno npr. $x_2=t$.

$$\begin{aligned}
2x_1+3x_2+2x_3+2x_4&=2\\
-x_3&=1\\
(\lambda-8)x_4&=0
\end{aligned}$$

$x_4=0$

$x_3=-1$

$2x_1+3t-2=2$

$2x_1=4-3t$

$x_1=2-\frac{3}{2}t$

Rješenje sistema (za $\lambda\neq 8$) je $(2-\frac{3}{2}t, t, -1, 0)$ gdje su $t\in\mathbb{R}$.

Riješiti sistem jednačina za razne vrijednosti parametra $\lambda\in\mathbb{R}$:

$$\begin{aligned}
\lambda x_1-4x_2+9x_3+10x_4&=11\\
2x_1-x_2+3x_3+4x_4&=5\\
4x_1-2x_2+5x_3+6x_4&=7\\
6x_1-3x_2+7x_3+8x_4&=9
\end{aligned}$$

Rj. Sistem ćemo riješiti Kroneker-Kapelijevom metodom:

$$\overline{A}=[A\,|\,b]=\left[\begin{array}{cccc|c} \lambda & -4 & 9 & 10 & 11\\ 2 & -1 & 3 & 4 & 5\\ 4 & -2 & 5 & 6 & 7\\ 6 & -3 & 7 & 8 & 9\end{array}\right] \underset{I_v\leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 6 & -3 & 7 & 8 & 9\\ 2 & -1 & 3 & 4 & 5\\ 4 & -2 & 5 & 6 & 7\\ \lambda & -4 & 9 & 10 & 11\end{array}\right] \underset{II_v\leftrightarrow I_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 2 & -1 & 3 & 4 & 5\\ 6 & -3 & 7 & 8 & 9\\ 4 & -2 & 5 & 6 & 7\\ \lambda & -4 & 9 & 10 & 11\end{array}\right] \underset{I_k\leftrightarrow IV_k}{\sim} \begin{array}{c}\begin{array}{cccc} x_4 & x_2 & x_3 & x_1 \end{array}\\ \left[\begin{array}{cccc|c} 4 & -1 & 3 & 2 & 5\\ 8 & -3 & 7 & 6 & 9\\ 6 & -2 & 5 & 4 & 7\\ 10 & -4 & 9 & \lambda & 11\end{array}\right]\end{array} \underset{I_k\leftrightarrow II_k}{\sim} \begin{array}{c}\begin{array}{cccc} x_2 & x_4 & x_3 & x_1 \end{array}\\ \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5\\ -3 & 8 & 7 & 6 & 9\\ -2 & 6 & 5 & 4 & 7\\ -4 & 10 & 9 & \lambda & 11\end{array}\right]\end{array}$$

$$\underset{\substack{II_v-I_v\cdot 3\\ III_v-I_v\cdot 2\\ IV_v-I_v\cdot 4}}{\sim} \left[\begin{array}{cccc|c} -1 & 4 & 3 & 2 & 5\\ 0 & -4 & -2 & 0 & -6\\ 0 & -2 & -1 & 0 & -3\\ 0 & -6 & -3 & \lambda-8 & -9\end{array}\right] \underset{II_k\leftrightarrow IV_k}{\sim} \begin{array}{c}\begin{array}{cccc} x_2 & x_1 & x_3 & x_4 \end{array}\\ \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5\\ 0 & 0 & -2 & -4 & -6\\ 0 & 0 & -1 & -2 & -3\\ 0 & \lambda-8 & -3 & -6 & -9\end{array}\right]\end{array} \underset{III_v\leftrightarrow II_v}{\sim} \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5\\ 0 & 0 & -1 & -2 & -3\\ 0 & 0 & -2 & -4 & -6\\ 0 & \lambda-8 & -3 & -6 & -9\end{array}\right]$$

$$\underset{\substack{III_v-II_v\cdot 2\\ IV_v-II_v\cdot 3}}{\sim} \begin{array}{c}\begin{array}{cccc} x_2 & x_1 & x_3 & x_4 \end{array}\\ \left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5\\ 0 & 0 & -1 & -2 & -3\\ 0 & 0 & 0 & 0 & 0\\ 0 & \lambda-8 & 0 & 0 & 0\end{array}\right]\end{array}$$

a) Za $\lambda=8$ imamo $\operatorname{rang} A=\operatorname{rang}\overline{A}=2<4$ pa prema Kroneker-Kapelijevoj teoremi sistem ima ∞ mnogo rješenja. 2. promjenjive uzimamo proizvoljno npr. $x_4=t$ $x_1=s$

$$\begin{aligned}
-x_3-2x_4&=-3\\
-x_2+2x_1+3x_3+4x_4&=5
\end{aligned}$$

$x_3=3-2t$

$-x_2+2s+3(3-2t)+4t=5$

$x_2=2s+9-6t+4t-5$

$x_2=2s-2t+4$

Za $\lambda=8$ rješenje sistema je $(s, 2s-2t+4, 3-2t, t)$ $t,s\in\mathbb{R}$

b) Za $\lambda\neq 8$ imamo $\operatorname{rang} A=\operatorname{rang}\overline{A}=3<4$ pa prema Kroneker-Kapelijevom teoremu sistem ima ∞ mnogo rješenja.

1. (jednu) promjenjivu uzimamo proizvoljno npr. $x_4=t$

$$\begin{aligned}
(\lambda-8)x_1&=0\\
-x_3-2x_4&=-3\\
-x_2+2x_1+3x_3+4x_4&=5
\end{aligned}$$

$x_1=0$

$x_3=3-2t$

$-x_2+3(3-2t)+4t=5$

$x_2=9-6t+4t-5=-2t+4$

Za $\lambda\neq 8$ rješenje sistema je $(0, 4-2t, 3-2t, t)$.

Cramerovo pravilo (metoda determinanti)

Rješavamo sistem oblika $A\cdot x = b$ gdje je $A=[a_{ij}]_{n\times n}$, $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$

$b=\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$. D_k determinanta koja se dobije od D ($D=\det A$) kada se umjesto k-te kolone u D stave slobodni članovi $\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$.

a) za $D\neq 0$ sistem ima jedinstveno rješenje $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$, $z=\frac{D_z}{D}$

b) za $D=0$ i ($D_x\neq 0$ ili $D_y\neq 0$ ili $D_z\neq 0$) sistem nema nijedno rješenje

c) za $D=D_x=D_y=D_z$ ne možemo ništa zaključiti (sistem može imati ∞ mnogo rješenja ili nemati nijedno rješenje) (potrebna su dalja ispitivanja)

) Metodom determinanti riješiti sistem jednačina

$$2x-y-z=4$$
$$3x+4y-2z=11$$
$$3x-2y+4z=11.$$

Rj.

$$D=\begin{vmatrix}2&-1&-1\\3&4&-2\\3&-2&4\end{vmatrix}\overset{II_v+I_v\cdot(-2)}{\underset{III_v+I_v\cdot 4}{=}}\begin{vmatrix}2&-1&-1\\-1&6&0\\11&-6&0\end{vmatrix}=(-1)\begin{vmatrix}-1&6\\11&-6\end{vmatrix}=-(6-66)=60$$

$$D_x=\begin{vmatrix}4&-1&-1\\11&4&-2\\11&-2&4\end{vmatrix}\overset{II_v-I_v\cdot 2}{\underset{III_v+I_v\cdot 4}{=}}\begin{vmatrix}4&-1&-1\\3&6&0\\27&-6&0\end{vmatrix}=(-1)\begin{vmatrix}3&6\\27&-6\end{vmatrix}=-(-18-162)=180$$

$$D_y=\begin{vmatrix}2&4&-1\\3&11&-2\\3&11&4\end{vmatrix}\overset{I_k+III_k\cdot 2}{\underset{II_k+III_k\cdot 4}{=}}\begin{vmatrix}0&0&-1\\-1&3&-2\\11&27&4\end{vmatrix}=(-1)\begin{vmatrix}-1&3\\11&27\end{vmatrix}=-(-27-33)=60$$

$$D_z=\begin{vmatrix}2&-1&4\\3&4&11\\3&-2&11\end{vmatrix}\overset{II_v+I_v\cdot 4}{\underset{III_v-I_v\cdot 2}{=}}\begin{vmatrix}2&-1&4\\11&0&27\\-1&0&3\end{vmatrix}=\begin{vmatrix}11&27\\-1&3\end{vmatrix}=3\begin{vmatrix}11&9\\-1&1\end{vmatrix}=3(11+9)=60$$

$$x=\frac{D_x}{D}=\frac{180}{60}=3;\quad y=\frac{D_y}{D}=\frac{60}{60}=1;\quad z=\frac{D_z}{D}=\frac{60}{60}=1$$

Rješenje sistema je $x=3$, $y=1$ i $z=1$

)^v Metodom determinanti riješiti sistem jednačina:

$$2x+4y-5z=-5$$
$$-x-y+z=0$$
$$2x+y-z=1.$$

Rj. $x=1$, $y=2$, $z=3$

) Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$(\lambda-2)x-3y+2z=1$$
$$3x-3y+(\lambda-3)z=1$$
$$x-y+2z=-1.$$

Rj.

$$D=\begin{vmatrix}\lambda-2&-3&2\\3&-3&\lambda-3\\1&-1&2\end{vmatrix}\overset{I_k+II_k}{\underset{III_k+II_k\cdot 2}{=}}\begin{vmatrix}\lambda-5&-3&-4\\0&-3&\lambda-9\\0&-1&0\end{vmatrix}=(\lambda-5)\begin{vmatrix}-3&\lambda-9\\-1&0\end{vmatrix}=(\lambda-5)(\lambda-9)$$

$$D_x=\begin{vmatrix}1&-3&2\\1&-3&\lambda-3\\-1&-1&2\end{vmatrix}\overset{I_v+III_v}{\underset{II_v+III_v}{=}}\begin{vmatrix}0&-4&4\\0&-4&\lambda-1\\-1&-1&2\end{vmatrix}=(-1)\begin{vmatrix}-4&4\\-4&\lambda-1\end{vmatrix}=(-1)(-4)\begin{vmatrix}1&4\\1&\lambda-1\end{vmatrix}=4(\lambda-5)$$

$$D_y=\begin{vmatrix}\lambda-2&1&2\\3&1&\lambda-3\\1&-1&2\end{vmatrix}\overset{I_v+III_v}{\underset{II_v+III_v}{=}}\begin{vmatrix}\lambda-1&0&4\\4&0&\lambda-1\\1&-1&2\end{vmatrix}=\begin{vmatrix}\lambda-1&4\\4&\lambda-1\end{vmatrix}=(\lambda-1)^2-4=(\lambda-1-4)(\lambda-1+4)=(\lambda-5)(\lambda+3)$$

$$D_z=\begin{vmatrix}\lambda-2&-3&1\\3&-3&1\\1&-1&-1\end{vmatrix}\overset{I_k+II_k}{=}\begin{vmatrix}\lambda-5&-3&1\\0&-3&1\\0&-1&-1\end{vmatrix}=(\lambda-5)\begin{vmatrix}-3&1\\-1&-1\end{vmatrix}=4(\lambda-5)$$

Diskusija

1° $\lambda\neq 5$ i $\lambda\neq 9$ ($D\neq 0$) Sistem ima jedinstveno rješenje

$$x=\frac{D_x}{D}=\frac{4(\lambda-5)}{(\lambda-5)(\lambda-9)}=\frac{4}{\lambda-9},\quad y=\frac{D_y}{D}=\frac{\lambda+3}{\lambda-9},\quad z=\frac{D_z}{D}=\frac{4}{\lambda-9}$$

2° $\lambda=9$

$D=0$, $D_x\neq 0$ $\Rightarrow$ sistem nema rješenja

3° $\lambda=5$ $\Rightarrow$ $D=D_x=D_y=D_z=0$ na osnovu Cramerovog pravila ne možemo ništa zaključiti. A trebno je uraditi sistem na drugi način.

za $\lambda=5$ sistem postaje

$$3x-3y+2z=1 \quad (1)$$
$$3x-3y+2z=1 \quad (2)$$
$$x-y+2z=-1 \quad (3)$$

(1) = (2)

(2) − (3): $2x-2y=2$

$x=y+1$

$x-y+2z=-1$

$y+1-y+2z=-1$

$2z=-2$

$z=-1$

sistem ima beskonačno mnogo rješenja koji su oblika $(t+1,\ t,\ -1)$, $t\in\mathbb{R}$

)^v Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra λ:

$$(\lambda+4)x+y+z=2$$
$$x+y+z=\lambda+5$$
$$3x+3y+(\lambda+7)z=3$$

Rj.

$D=(\lambda+4)(\lambda+3)$

$D_x=-(\lambda+4)(\lambda+3)$

$D_y=(\lambda+3)(\lambda+4)(\lambda+8)$

$D_z=-3(\lambda+3)(\lambda+4)$

$t\in\mathbb{R}$

$(t, 5-t, -3)$

$(-1, 2-s, s)$

$s\in\mathbb{R}$

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ

$$x+y+z=4$$
$$x+\lambda y+z=3$$
$$x+2\lambda y+z=4 \;.$$

Rj. Sistem rješavamo Cramerovom metodom

$$D=\begin{vmatrix}1&1&1\\1&\lambda&1\\1&2\lambda&1\end{vmatrix} \underset{=\!=}{\overset{II_v-III_v}{}} \begin{vmatrix}1&1&1\\0&-\lambda&0\\1&2\lambda&1\end{vmatrix} = -\lambda\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}4&1&1\\3&\lambda&1\\4&2\lambda&1\end{vmatrix} \underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}} \begin{vmatrix}1&1-\lambda&0\\3&\lambda&1\\1&\lambda&0\end{vmatrix} = -\begin{vmatrix}1&1-\lambda\\1&\lambda\end{vmatrix} = -(\lambda-(1-\lambda)) = 1-\lambda-\lambda = 1-2\lambda$$

$$D_y=\begin{vmatrix}1&4&1\\1&3&1\\1&4&1\end{vmatrix} \overset{III_k-I_k}{=\!=} \begin{vmatrix}1&4&0\\1&3&0\\1&4&0\end{vmatrix}=0$$

$$D_z=\begin{vmatrix}1&1&4\\1&\lambda&3\\1&2\lambda&4\end{vmatrix} \underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}} \begin{vmatrix}0&1-\lambda&1\\1&\lambda&3\\0&\lambda&1\end{vmatrix} = -\begin{vmatrix}1-\lambda&1\\\lambda&1\end{vmatrix} = -(1-\lambda-\lambda) = 2\lambda-1$$

Kako je $D=0$ to sistem može da ima beskonačno mnogo rješenja ili da nema rješenja.

1° $\lambda=\frac{1}{2}$

$D=0,\ D_x=0,\ D_y=0,\ D_z=0$

Sistem ćemo riješiti Gauss-ovom metodom

$$x+y+z=4$$
$$x+\tfrac{1}{2}y+z=3 \quad /\cdot 2$$
$$x+y+z=4$$

$$x+y+z=4 \quad (1)$$
$$2x+y+2z=6 \quad (2)$$
$$(2)-(1):\quad x+z=2$$
$$x=2-z$$

$$x+y+z=4$$
$$2-z+y+z=4$$
$$y=2$$

Za $\lambda=\frac{1}{2}$ sistem ima ∞ mnogo rješenja koja su oblika $(2-t,2,t)$ gdje je $t\in\mathbb{R}$.

2° $\lambda\neq\frac{1}{2}$

$D=0,\ D_x\neq 0 \Rightarrow$ sistem za $\lambda\neq\frac{1}{2}$ nema rješenja

(#) Odrediti vrijednost parametra k tako da sistem

$$8z-3x-6y=kx$$
$$2x+y+4z=ky$$
$$4x+3y+z=kz$$

ima beskonačno mnogo rješenja. Zatim naći ta rješenja za najveću dobijenu vrijednost parametra k.

Rj. Nepoznate sa desne strane prebacimo na lijevu i grupirajmo vrijednosti uz x, y i z.

$$(-3-k)x-6y+8z=0$$
$$2x+(1-k)y+4z=0$$
$$4x+3y+(1-k)z=0$$

Ovo je homogeni sistem linearnih jednačina. Trivijalno rješenje je $(0,0,0)$. Sistem ima beskonačno mnogo rješenja ako je $D=0$.

$$\begin{vmatrix}-3-k&-6&8\\2&1-k&4\\4&3&1-k\end{vmatrix}=0$$

I_k+III_k:
$$\begin{vmatrix}5-k&-6&8\\6&1-k&4\\5-k&3&1-k\end{vmatrix}=0$$

I_v-III_v:
$$\begin{vmatrix}0&-9&7+k\\6&1-k&4\\5-k&3&1-k\end{vmatrix}=0$$

$$6\Big[\overbrace{(-9)(1-k)}^{-9+9k}-\overbrace{3(7+k)}^{-21-3k}\Big]+(5-k)\Big[(-9)\cdot 4-\overbrace{(7+k)(1-k)}^{\substack{7-6k-k^2\\7-7k+k-k^2}}\Big]=0$$

$$(-6)(6k-30)+(5-k)(\overbrace{-36-7}^{-43}+6k+k^2)=0$$

$$-36k+180+(-215)+30k+5k^2+43k-6k^2-k^3=0$$

$$-k^3-k^2+37k-35=0 \quad /\cdot(-1)$$

$$k^3+k^2-37k+35=0$$
$$k^3-k^2+2k^2-2k-35k+35=0$$
$$k^2(1-k)+2k(k-1)-35(k-1)=0$$

$$(k-1)(k^2+2k-35)=0$$
$$(k-1)(k+7)(k-5)=0$$
$$k_1=1,\ k_2=-7,\ k_3=5$$

Za $k=5$ imamo:

$$8x+6y-8z=0 \quad \dots(1)$$
$$2x-4y+4z=0 \quad \dots(2)$$
$$4x+3y-4z=0 \quad \dots(3)$$

$(1)\equiv(3)$ jer se (3) dobija dijeljenjem (1) sa 2.

$(2)+(3):\quad 6x-y=0$
$\Rightarrow y=6x$

$(2)\Rightarrow 2x-24x+4z=0$
$\therefore\ 4z=22x$
$z=\frac{11x}{2}$

Za $k=5$ sistem ima rješenje $(t,\,6t,\,\frac{11t}{2})$ gdje je $t\in\mathbb{R}$ proizvoljno.

Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} x - y - \lambda z &= 1 \\ (\lambda+1)y + (\lambda-1)z &= 0 \\ (\lambda+1)x - (\lambda+1)z &= 1 . \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & -(\lambda+1) \end{vmatrix} \underset{=}{\overset{III_k + I_k}{=}} \begin{vmatrix} 1 & -1 & 1-\lambda \\ 0 & \lambda+1 & \lambda-1 \\ \lambda+1 & 0 & 0 \end{vmatrix} = (\lambda+1) \begin{vmatrix} -1 & -(\lambda-1) \\ \lambda+1 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda+1)(\lambda-1) \begin{vmatrix} -1 & -1 \\ \lambda+1 & 1 \end{vmatrix} = \lambda(\lambda-1)(\lambda+1)$$

$$D_x = \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} \underset{=}{\overset{III_v - I_v}{=}} \begin{vmatrix} 1 & -1 & -\lambda \\ 0 & \lambda+1 & \lambda-1 \\ 0 & 1 & -1 \end{vmatrix} = \overset{-1+\lambda+1}{\begin{vmatrix} \lambda+1 & \lambda-1 \\ 1 & -1 \end{vmatrix}} = \lambda-1-\lambda+1 = -2\lambda$$

$$D_y = \begin{vmatrix} 1 & 1 & -\lambda \\ 0 & 0 & \lambda-1 \\ \lambda+1 & 1 & -(\lambda+1) \end{vmatrix} = -(\lambda-1) \begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -(\lambda-1)(1-\lambda-1) = \lambda(\lambda-1)$$

$$D_z = \begin{vmatrix} 1 & -1 & 1 \\ 0 & \lambda+1 & 0 \\ \lambda+1 & 0 & 1 \end{vmatrix} = (\lambda+1) \begin{vmatrix} 1 & 1 \\ \lambda+1 & 1 \end{vmatrix} = -\lambda(\lambda+1)$$

$D=0$ akko $\lambda=0$ ili $\lambda=1$ ili $\lambda=-1$

Diskusija

1° $\lambda \neq 0$; $\lambda \neq 1$; $\lambda \neq -1$ sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = \frac{-2\lambda}{\lambda(\lambda-1)(\lambda+1)} = \frac{-2}{(\lambda-1)(\lambda+1)}, \quad y = \frac{D_y}{D} = \frac{1}{\lambda+1}, \quad z = \frac{D_z}{D} = \frac{-1}{\lambda+1}$$

2° $\lambda=1$, $D=0$, $D_x \neq 0$ $\Rightarrow$ sistem nema rješenja.

3° $\lambda=-1$, $D=0$, $D_x \neq 0$ $\Rightarrow$ sistem nema rješenja

4° $\lambda=0$, $D=D_x=D_y=D_z=0$ iz ovoga ne možemo ništa zaključiti

Za $\lambda=0$ sistem postaje

$$\begin{aligned} x - y &= 1 \quad (1) \\ y - z &= 0 \quad (2) \\ x - z &= 1 \quad (3) \end{aligned}$$

(1): $x - y = 1$
(2)−(3): $-x + y = -1$
$x = y+1$

$x - z = 1$
$-z = -(y+1)+1$
$-z = -y$
$z = y$

Sistem ima ∞ mnogo rješenja $(t+1, t, t)$, $t \in \mathbb{R}$

Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra a:

$$\begin{aligned} x + y - z &= 0 \\ x - y + az &= 1 \\ -x - 3y + (a+2)z &= a^2 . \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & a \\ -1 & -3 & a+2 \end{vmatrix} \underset{II_k + III_k}{\overset{I_k + III_k}{=}} \begin{vmatrix} 0 & 0 & -1 \\ a+1 & a-1 & a \\ a+1 & a-1 & a+2 \end{vmatrix} = (-1) \begin{vmatrix} a+1 & a-1 \\ a+1 & a-1 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & a \\ a^2 & -3 & a+2 \end{vmatrix} \underset{=}{\overset{II_k + III_k}{=}} \begin{vmatrix} 0 & 0 & -1 \\ 1 & a-1 & a \\ a^2 & a-1 & a+2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & a-1 \\ a^2 & a-1 \end{vmatrix} = (-1)(a-1) \begin{vmatrix} 1 & 1 \\ a^2 & 1 \end{vmatrix}$$

$$= (-1)(a-1)(1-a^2) = (a-1)(a^2-1) = (a-1)^2(a+1)$$

$$D_y = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & a \\ -1 & a^2 & a+2 \end{vmatrix} \underset{=}{\overset{I_k + III_k}{=}} \begin{vmatrix} 0 & 0 & -1 \\ a+1 & 1 & a \\ a+1 & a^2 & a+2 \end{vmatrix} = (-1) \begin{vmatrix} a+1 & 1 \\ a+1 & a^2 \end{vmatrix} = (-1)(a+1) \begin{vmatrix} 1 & 1 \\ 1 & a^2 \end{vmatrix} = (-1)(a+1)(a^2-1)$$

$$= (-1)(a-1)(a+1)^2$$

$$D_z = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & -3 & a^2 \end{vmatrix} \underset{=}{\overset{I_k - II_k}{=}} \begin{vmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & -3 & a^2 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 1 \\ 2 & a^2 \end{vmatrix} = (-1)(2a^2-2) = (-2)(a+1)(a-1)$$

Diskusija
$D=0$ $\forall a \in \mathbb{R}$

1° $a \neq 1$; $a \neq -1$
$D=0$; $D_x \neq 0$ sistem nema rješenja

2° $a=1$
$D=D_x=D_y=D_z=0$, sistem postaje

$$\begin{aligned} x + y - z &= 0 \quad (1) \\ x - y + z &= 1 \quad (2) \\ -x - 3y + 3z &= 1 \quad (3) \end{aligned}$$

(1)+(3): $-2y + 2z = 1$
(2)+(3): $-4y + 4z = 2$
$2z = 2y+1$
$z = y + \frac{1}{2}$
$x = z - y$
$x = \frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $\left(\frac{1}{2}, t, t+\frac{1}{2}\right)$ gdje je $t \in \mathbb{R}$.

3° $a=-1$
$D=D_x=D_y=D_z=0$, sistem postaje

$$\begin{aligned} x + y - z &= 0 \quad I \\ x - y - z &= 1 \quad II \\ -x - 3y + z &= 1 \quad III \end{aligned}$$

(I)+(III): $-2y = 1$
(II)+(III): $-4y = 2$
$y = -\frac{1}{2}$

(I)+(II): $2x - 2z = 1$
(III)−3·(II): $-4x + 4z = 2$
$2x = 2z+1$
$x = z + \frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $\left(t+\frac{1}{2}, -\frac{1}{2}, t\right)$, $t \in \mathbb{Z}$

Diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} 2x - \lambda y + 2z &= 1 \\ x + y + 2z &= 0 \\ -x + (-\lambda-3)y - 4z &= \lambda \end{aligned}$$

Rj. Sistem ćemo riješiti Cramerovim pravilima.

$$D = \begin{vmatrix} 2 & -\lambda & 2 \\ 1 & 1 & 2 \\ -1 & -\lambda-3 & -4 \end{vmatrix} \underset{III_k - II_k \cdot 2}{\overset{I_k - II_k}{=}} \begin{vmatrix} 2+\lambda & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} \lambda+2 & 2\lambda+2 \\ \lambda+2 & 2\lambda+2 \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 2\lambda+2 \\ 1 & 2\lambda+2 \end{vmatrix} = 0$$

$$D_x = \begin{vmatrix} 1 & -\lambda & 2 \\ 0 & 1 & 2 \\ \lambda & -\lambda-3 & -4 \end{vmatrix} \overset{III_k - II_k \cdot 2}{=} \begin{vmatrix} 1 & -\lambda & 2\lambda+2 \\ 0 & 1 & 0 \\ \lambda & -\lambda-3 & 2\lambda+2 \end{vmatrix} = \begin{vmatrix} 1 & 2\lambda+2 \\ \lambda & 2\lambda+2 \end{vmatrix} = (2\lambda+2)\begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = (2\lambda+2)(1-\lambda) = 2(\lambda+1)(1-\lambda)$$

$$D_y = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ -1 & \lambda & -4 \end{vmatrix} \overset{III_k - I_k \cdot 2}{=} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ -1 & \lambda & -2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -2 \\ \lambda & -2 \end{vmatrix} = (-1)(-2)\begin{vmatrix} 1 & 1 \\ \lambda & 1 \end{vmatrix} = 2(1-\lambda)$$

$$D_z = \begin{vmatrix} 2 & -\lambda & 1 \\ 1 & 1 & 0 \\ -1 & -\lambda-3 & \lambda \end{vmatrix} \overset{I_k - II_k}{=} \begin{vmatrix} 2+\lambda & -\lambda & 1 \\ 0 & 1 & 0 \\ \lambda+2 & -\lambda-3 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 \\ \lambda+2 & \lambda \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 1 \\ 1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda-1)$$

Diskusija:

$D=0$, $D_x = 2(1+\lambda)(1-\lambda)$, $D_y = 2(1-\lambda)$, $D_z = (\lambda+2)(\lambda-1)$

1° $\lambda \neq -1$; $\lambda \neq 1$; $\lambda \neq -2$
imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

2° $\lambda = -2$ imamo $D=0$; $D_x \neq 0$ sistem nema rješenja

3° $\lambda = -1$ imamo $D=0$, $D_x = 0$, $D_y \neq 0$ sistem nema rješenja

4° $\lambda = 1$ imamo $D = D_x = D_y = D_z = 0$ sistem je potrebno ispitati na drugi način.

Za $\lambda=1$ sistem postaje

$$\begin{aligned} 2x - y + 2z &= 1 \quad /\cdot 4 \\ x + y + 2z &= 0 \quad /\cdot 4 \\ -x - 4y - 4z &= 1 \end{aligned}$$

$$\begin{aligned} 8x - 4y + 8z &= 4 \quad (1) \\ 4x + 4y + 8z &= 0 \quad (2) \\ -x - 4y - 4z &= 1 \quad (3) \end{aligned}$$

$(1)+(2)$: $12x + 16z = 4$

$(3)+(2)$: $3x + 4z = 1$

$3x = 1 - 4z$

$x = \dfrac{1-4z}{3}$

$y = -x - 2z$

$y = \dfrac{4z-1}{3} - \dfrac{6z}{3}$

Sistem ima ∞ mnogo rješenja, oblika $\left(\dfrac{1-4t}{3}, \dfrac{-2t-1}{3}, t\right)$, $t \in \mathbb{R}$

Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra

$$\begin{aligned} x + y + bz &= 1 - b \\ x - by - z &= 2 \\ bx - y + z &= 2b \end{aligned}$$

Rj. Rješavamo sistem Cramerovom metodom

$$D = \begin{vmatrix} 1 & 1 & b \\ 1 & -b & -1 \\ b & -1 & 1 \end{vmatrix} \overset{I_k + III_k}{=} \begin{vmatrix} b+1 & 1 & b \\ 0 & -b & -1 \\ b+1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 1 & b \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{I_v - III_v}{=}$$

$$= (b+1)\begin{vmatrix} 0 & 2 & b-1 \\ 0 & -b & -1 \\ 1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 2 & b-1 \\ -b & -1 \end{vmatrix} = (b+1)\overbrace{\left[-2+(b^2-b)\right]}^{b^2-b-2} = (b+1)(b+1)(b-2)$$

$$D_x = \begin{vmatrix} 1-b & 1 & b \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} \overset{I_v + III_v}{=} \begin{vmatrix} b+1 & 0 & b+1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 0 & 1 \\ 2 & -b & -1 \\ 2b & -1 & 1 \end{vmatrix} =$$

$$\overset{I_k - III_k}{=} (b+1)\begin{vmatrix} 0 & 0 & 1 \\ 3 & -b & -1 \\ 2b-1 & -1 & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 3 & -b \\ 2b-1 & -1 \end{vmatrix} = (b+1)\overbrace{(-3+2b^2-b)}^{2b^2-b-3} = (b+1)\cdot 2\left(b-\frac{3}{2}\right)(b+1) = (b+1)(2b-3)(b+1)$$

$D = 1 + 24 = 25$, $b_{1,2} = \dfrac{1 \pm 5}{4}$

$$D_y = \begin{vmatrix} 1 & 1-b & b \\ 1 & 2 & -1 \\ b & 2b & 1 \end{vmatrix} \overset{I_k + III_k}{=} \begin{vmatrix} b+1 & 1-b & b \\ 0 & 2 & -1 \\ b+1 & 2b & 1 \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 1 & 2b & 1 \end{vmatrix} \overset{III_v - I_v}{=}$$

$$= (b+1)\begin{vmatrix} 1 & 1-b & b \\ 0 & 2 & -1 \\ 0 & 3b-1 & 1-b \end{vmatrix} = (b+1)\begin{vmatrix} 2 & -1 \\ 3b-1 & 1-b \end{vmatrix} = (b+1)(2-2b+3b-1) = (b+1)(b+1)$$

$$D_z = \begin{vmatrix} 1 & 1 & 1-b \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \overset{I_v + III_v}{=} \begin{vmatrix} b+1 & 0 & b+1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} = (b+1)\begin{vmatrix} 1 & 0 & 1 \\ 1 & -b & 2 \\ b & -1 & 2b \end{vmatrix} \overset{I_k - III_k}{=}$$

$$= (b+1)\begin{vmatrix} 0 & 0 & 1 \\ -1 & -b & 2 \\ -b & -1 & 2b \end{vmatrix} = (b+1)\begin{vmatrix} -1 & -b \\ -b & -1 \end{vmatrix} = (b+1)(1-b^2) = -(b+1)(b^2-1) = -(b+1)(b-1)(b+1)$$

Diskusija: a) $D \neq 0$ tj. $b \neq -1$; $b \neq 2$

sistem ima jedinstveno rješenje $x = \dfrac{D_x}{D} = \dfrac{(2b-3)(b+1)^2}{(b+1)^2(b-2)} = \dfrac{2b-3}{b-2}$

$y = \dfrac{D_y}{D} = \dfrac{(b+1)^2}{(b+1)^2(b-2)} = \dfrac{1}{b-2}$; $z = \dfrac{D_z}{D} = \dfrac{-(b-1)(b+1)^2}{(b-2)(b+1)^2} = -\dfrac{b-1}{b-2}$

b) $b=-1 \Rightarrow D=D_x=D_y=D_z=0$ sistem trebamo riješiti na drugi način

Za $b=-1$ sistem postaje
$$\begin{aligned} x+y-z&=2\\ x+y-z&=2\\ -x-y+z&=-2 \quad /\cdot(-1) \end{aligned}$$

Sve tri jednačine su iste $\Rightarrow$ Sistem ima ∞ mnogo rješenja. Ako uzmemo $x=t$, $y=s$ Rješenja sistema su $(t, s, t+s-2)$ ← dvije promjenjive uzimamo proizvoljno

c) $b=2 \Rightarrow D=0$, $D_x=9\neq 0 \Rightarrow$
Sistem za $b=2$ nema rješenja

Homogeni sistemi linearnih jednačina

Homogeni sistem linearnih jednačina je oblika $A\cdot x=0$ gdje je
$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{bmatrix}_{m\times n}, \quad x=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}_{n\times 1} \quad i \quad 0=\begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}_{m\times 1}$$

Teorema: Homogeni sistem ima netrivijalna rješenja akko je $D=0$ ($\det A=0$).

1. Riješiti homogeni sistem jednačina
$$\begin{aligned} x_1+x_2+x_3&=0 \quad (1)\\ 3x_1+x_2-x_3&=0 \quad (2)\\ 2x_1+x_2&=0.\end{aligned}$$

Rj. $(1)+(2)$
$$\begin{aligned} 4x_1+2x_2&=0\\ 2x_1+x_2&=0 \quad /\cdot 2\\ \hline 4x_1+2x_2&=0\\ 4x_1+2x_2&=0\\ \hline\end{aligned}$$

$4x_1+2x_2=0 \quad /:2$
$2x_1+x_2=0$
sistem ima ∞ mnogo rješenja
$x_2=-2x_1$
$x_1=t$, $x_2=-2t$, $t\in\mathbb{R}$
$t-2t+x_3=0$
$x_3=t$

Sistem ima beskonačno mnogo rješenja oblika $(t,-2t,t)$

2. Nađi λ tako da sistem
$$\begin{aligned} 3x+y+\lambda z&=0\\ 4x-8y+\lambda z&=0\\ 5x-3y+3z&=0\end{aligned}$$
ima netrivijalna rješenja pa nađi rješenja.

Rj.
$$D=\begin{vmatrix} 3 & 1 & \lambda\\ 4 & -8 & \lambda\\ 5 & -3 & 3\end{vmatrix} \underset{III_v+I_v\cdot 3}{\overset{II_v+I_v\cdot 8}{=\!=}} \begin{vmatrix} 3 & 1 & \lambda\\ 28 & 0 & 9\lambda\\ 14 & 0 & 3\lambda+3\end{vmatrix} = -\begin{vmatrix} 28 & 9\lambda\\ 14 & 3\lambda+3\end{vmatrix} = (-14)\cdot 3\begin{vmatrix} 2 & 3\lambda\\ 1 & \lambda+1\end{vmatrix} = -42(-\lambda+2)$$

Za $\lambda=2$ ($D=0$) u sistemu postoje netrivijalna rješenja.
Sistem sad izgleda:
$$\begin{aligned} 3x+y+2z&=0 \quad /\cdot 3\\ 4x-8y+2z&=0 \quad /\cdot 3\\ 5x-3y+3z&=0 \quad /\cdot 2\\ \hline\end{aligned} \qquad \begin{aligned} 9x+3y+6z&=0 \quad (1)\\ 12x-24y+6z&=0 \quad (2)\\ 10x-6y+6z&=0 \quad (3)\\ \hline\end{aligned}$$

$(3)-(1)$: $x-9y=0$
$(2)-(1)$: $3x-27y=0 \quad /:3$
$x-9y=0$
$x=9y$, $z=-14y$ postoji ∞ mnogo rješenja
$(9t, t, -14t)$, $t\in\mathbb{R}$ su rješenja sistema

3. Za koje vrijednosti λ sistem ima netrivijalna rješenja
$$\begin{aligned} \lambda x_1+x_2+x_3+x_4&=0\\ x_1+\lambda x_2+x_3+x_4&=0\\ x_1+x_2+\lambda x_3+x_4&=0\\ x_1+x_2+x_3+\lambda x_4&=0.\end{aligned}$$

Rj. za $\lambda=1$ ili $\lambda=-3$

Vektorski prostor

Vektorski prostor je svaki neprazan skup $V=\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n, \dots\}$ u kojem su definisane računske operacije sabiranja vektora i množenje vektora sa skalarom na sljedeći način:

a) $\forall(\vec{a},\vec{b}\in V)$ $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (komutativnost sabiranja)

b) $\forall(\vec{a},\vec{b},\vec{c}\in V)$ $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (asocijativnost sabiranja)

c) $\forall(\vec{a}\in V)$ $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$ (nula je neutralni elem. za sabiranje)

d) $\forall(\vec{a}\in V)$ $\exists(-\vec{a}\in V)$ $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$ (suprotni element)

e) $\forall(\vec{a},\vec{b}\in V)$ $\forall(\alpha\in R)$ $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$ (distributivnost množenja prema sabir.)

f) $\forall(\vec{a}\in V)$ $\forall(\alpha,\beta\in R)$ $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$ (distributivnost sabiranja prema množenju)

g) $\forall(\vec{a}\in V)$ $\forall(\alpha,\beta\in R)$ $(\alpha\beta)\vec{a}=\alpha(\beta\vec{a})$

h) $\forall(\vec{a}\in V)$ $1\cdot\vec{a}=\vec{a}\cdot 1=\vec{a}$

Elemente $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n, \dots$ zovemo VEKTORI.

Za vektore $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ kažemo da su LINEARNO ZAVISNI ako postoje skalari $\alpha_1, \alpha_2, \dots, \alpha_n$ takvi da je $\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\dots+\alpha_n\vec{a}_n=\vec{0}$ i postoji skalar $\alpha_1, \alpha_2, \dots, \alpha_n$ koji nije jednak nuli.

Ako jednakost $\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\dots+\alpha_n\vec{a}_n=\vec{0}$ vrijedi samo u slučaju kada je $\alpha_1=\alpha_2=\dots=\alpha_n=0$ onda su vektori $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ LINEARNO NEZAVISNI.

Za vektor $\vec{a}$ kažemo da je LINEARNA KOMBINACIJA vektora $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ (ili kažemo da je RAZLOŽEN preko vektora $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$) ako postoje skalari $\alpha_1, \alpha_2, \dots, \alpha_n$ takvi da je $\vec{a}=\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\dots+\alpha_n\vec{a}_n$.

Ako su vektori $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ baza vektorskog prostora prethodnu jednakost možemo pisati kao $\vec{a}=(\alpha_1, \alpha_2, \dots, \alpha_n)$

Svaki skup linearno nezavisnih vektora iz V čini **BAZU** tog prostora. Broj elemenata baze vektorskog prostora čini DIMENZIJU tog prostora.

\# Vektor $\vec{a}=(1,0,1)$ izraziti kao linearnu kombinaciju vektora $\vec{b}=(2,1,0)$, $\vec{c}=(2,-1,1)$ i $\vec{d}=(0,2,0)$.

Rj. Želimo pronaći konstante β, γ i δ takve da

$$\vec{a}=\beta\vec{b}+\gamma\vec{c}+\delta\vec{d}$$

tj.

$$(1,0,1)=\beta(2,1,0)+\gamma(2,-1,1)+\delta(0,2,0)$$

$$\begin{aligned} 2\beta+2\gamma+0\delta&=1\\ \beta-\gamma+2\delta&=0\\ 0\beta+\gamma+0\delta&=1 \quad\Rightarrow\quad \gamma=1\\ \hline 2\beta+2\quad&=1 \quad\Rightarrow\quad 2\beta=-1\\ &\qquad\qquad\quad \beta=-\tfrac{1}{2}\end{aligned}$$

$$\beta-\gamma+2\delta=0 \quad\Rightarrow\quad -\frac{1}{2}-1+2\delta=0$$

$$2\delta=\frac{3}{2}$$

$$\delta=\frac{3}{4}$$

Prema tome $\vec{a}=-\frac{1}{2}\vec{b}+\vec{c}+\frac{3}{4}\vec{d}$

(vektor $\vec{a}$ izražen kao linearna kombinacija vektora $\vec{b}$, $\vec{c}$ i $\vec{d}$).

○ Ispitati linearnu zavisnost vektora $\vec{a}=(2,3,-4)$, $\vec{b}=(3,-2,0)$ i $\vec{c}=(0,1,1)$.

Rj: $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$\alpha(2,3,-4)+\beta(3,-2,0)+\gamma(0,1,1)=(0,0,0)$

$$\begin{aligned} 2\alpha+3\beta \quad &=0\\ 3\alpha-2\beta+\gamma&=0\\ -4\alpha \quad +\gamma&=0 \end{aligned}$$

$$\det M=\begin{vmatrix}2&3&0\\3&-2&1\\-4&0&1\end{vmatrix}\xlongequal{III_v-II_v}\begin{vmatrix}2&3&0\\3&-2&1\\-7&2&0\end{vmatrix}=$$

$$=(-1)\begin{vmatrix}2&3\\-7&2\end{vmatrix}=(-1)(4+21)=-25$$

$\det M\neq 0$

sistem ima samo trivijalna rješenja $((0,0,0))$

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno nezavisni.

○ Dokazati da su vektori $\vec{a}=(3,1,8)$, $\vec{b}=(3,4,5)$ i $\vec{c}=(2,3,3)$ linearno zavisni.

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$\alpha(3,1,8)+\beta(3,4,5)+\gamma(2,3,3)=(0,0,0)$

$$\begin{aligned} 3\alpha+3\beta+2\gamma&=0\\ \alpha+4\beta+3\gamma&=0\\ 8\alpha+5\beta+3\gamma&=0 \end{aligned}$$

$$\det M=\begin{vmatrix}3&3&2\\1&4&3\\8&5&3\end{vmatrix}\underset{III_v-II_v\cdot 8}{\xlongequal{I_v-II_v\cdot 3}}\begin{vmatrix}0&-9&-7\\1&4&3\\0&-27&-21\end{vmatrix}=(-1)\begin{vmatrix}-9&-7\\-27&-21\end{vmatrix}$$

$$=(-1)(-9)(-7)\begin{vmatrix}1&1\\3&3\end{vmatrix}=0$$

$\det M=0$

rang $M<3$

sistem ima netrivijalna rješenja

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno zavisni.

○ Diskutovati linearnu zavisnost vektora $\vec{a}=(3,-8,2)$, $\vec{b}=(7,6,5)$ i $\vec{c}=(5,2,6-\lambda)$ u zavisnosti od parametra λ.

Rj. $\det M=182-74\lambda$

1° $\lambda=\frac{182}{74}$ vektori linearno zavisni

2° $\lambda\neq\frac{182}{74}$ vektori linearno nezavisni

Dat je skup $\mathcal{B}=\left\{\begin{pmatrix}3\\-6\\-9\end{pmatrix},\begin{pmatrix}2\\-5\\-6\end{pmatrix},\begin{pmatrix}-1\\1\\5\end{pmatrix}\right\}$. Provjeriti da li je skup $\mathcal{B}$ linearno nezavisan. Da li je $\mathcal{B}$ baza vektorskog prostora $\mathbb{R}^3$. Zašto? Vektor $u=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ izraziti kao linearnu kombinaciju vektora iz baze $\mathcal{B}$ (drugim riječima odrediti koordinate vektora $u=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ u odnosu na bazu $\mathcal{B}$).

Rj-upute:

Skup $\mathcal{B}$ je linearno nezavisan akko jedino rješenje sistema

$$\alpha\begin{pmatrix}3\\-6\\-9\end{pmatrix}+\beta\begin{pmatrix}2\\-5\\-6\end{pmatrix}+\gamma\begin{pmatrix}-1\\1\\5\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$ po nepoznatim α,β i γ

je trivijalno rješenje $\alpha=\beta=\gamma=0$.

$$\begin{aligned} 3\alpha+2\beta-\gamma&=0\\ -6\alpha+(-5)\beta+\gamma&=0\\ -9\alpha-6\beta+5\gamma&=0 \end{aligned}$$

ovo je homogeni sistem (uvijek ima jedno rješenje)

$$D=\begin{vmatrix}3&2&-1\\-6&-5&1\\-9&-6&5\end{vmatrix}=-6\neq 0$$

$D\neq 0$ skup $\mathcal{B}$ je linearno nezavisan

$\mathcal{B}$ jest baza vektorskog prostora $\mathbb{R}^3$ zato što skup bilo koja tri linearno nezavisna vektora formiraju bazu od $\mathbb{R}^3$.

Koordinate vektora u u odnosu na bazu $\mathcal{B}$ su $\begin{pmatrix}2\\-2\\1\end{pmatrix}$, drugim riječima

$$u=2\begin{pmatrix}3\\-6\\-9\end{pmatrix}-2\begin{pmatrix}2\\-5\\-6\end{pmatrix}+\begin{pmatrix}-1\\1\\5\end{pmatrix}.$$

\# Vektor $v \in \mathbb{R}^3$ u odnosu na bazu $\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\6\end{pmatrix}, \begin{pmatrix}0\\2\\6\end{pmatrix}, \begin{pmatrix}0\\0\\-3\end{pmatrix} \right\}$ ima koordinate $\begin{pmatrix}4\\-1\\7\end{pmatrix}$. Odrediti koordinate vektora v u odnosu na bazu $\mathcal{B}' = \left\{ \begin{pmatrix}0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\2\\3\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$.

Rj.-upute.

Posmatrajmo baze $\mathcal{B}$ i $\mathcal{B}'$. Nije teško vidjeti da je

$$\left.\begin{aligned}
\begin{pmatrix}1\\2\\6\end{pmatrix} &= 1\cdot\begin{pmatrix}0\\0\\3\end{pmatrix} + 0\begin{pmatrix}0\\2\\3\end{pmatrix} + 1\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}\\
\begin{pmatrix}0\\2\\6\end{pmatrix} &= 1\cdot\begin{pmatrix}0\\0\\3\end{pmatrix} + 1\cdot\begin{pmatrix}0\\2\\3\end{pmatrix} + 0\begin{pmatrix}1\\2\\3\end{pmatrix}\\
\begin{pmatrix}0\\0\\-3\end{pmatrix} &= (-1)\begin{pmatrix}0\\0\\3\end{pmatrix} + 0\begin{pmatrix}0\\2\\3\end{pmatrix} + 0\begin{pmatrix}1\\2\\3\end{pmatrix}
\end{aligned}\right\} \ldots (*)$$

Kako su koordinate vektora v u odnosu na bazu $\mathcal{B}$ $\begin{pmatrix}4\\-1\\7\end{pmatrix}$ to znači da je $v = 4\begin{pmatrix}1\\2\\6\end{pmatrix} - 1\begin{pmatrix}0\\2\\6\end{pmatrix} + 7\begin{pmatrix}0\\0\\-3\end{pmatrix}$.

Prema $(*)$ imamo

$$4\begin{pmatrix}1\\2\\6\end{pmatrix} = 4\begin{pmatrix}0\\0\\3\end{pmatrix} + \qquad + 4\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$(-1)\begin{pmatrix}0\\2\\6\end{pmatrix} = (-1)\begin{pmatrix}0\\0\\3\end{pmatrix} + (-1)\begin{pmatrix}0\\2\\3\end{pmatrix}$$

$$7\begin{pmatrix}0\\0\\-3\end{pmatrix} = (-7)\begin{pmatrix}0\\0\\3\end{pmatrix}$$

Prema tome $v = (-4)\begin{pmatrix}0\\0\\3\end{pmatrix} - 1\begin{pmatrix}0\\2\\3\end{pmatrix} + 4\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Koordinate vektora v u odnosu na bazu $\mathcal{B}'$ su $\begin{pmatrix}-4\\-1\\4\end{pmatrix}$.

\# Odrediti sve vrijednosti parametra m tako da vektori $\vec{a} = \begin{pmatrix}m-2\\1\\2\end{pmatrix}$, $\vec{b} = \begin{pmatrix}m-2\\m-2\\3\end{pmatrix}$, $\vec{c} = \begin{pmatrix}m-2\\1\\m-2\end{pmatrix}$ nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rj.-uputa.

Vektori $\vec{a}, \vec{b}$ i $\vec{c}$ neće činiti bazu vektorskog prostora $\mathbb{R}^3$ ako su linearno zavisni, a oni su linearno zavisni ako postoje brojevi α, β i γ (ne svi nula) taki da $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$,

$$\alpha\begin{pmatrix}m-2\\1\\2\end{pmatrix} + \beta\begin{pmatrix}m-2\\m-2\\3\end{pmatrix} + \gamma\begin{pmatrix}m-2\\1\\m-2\end{pmatrix} = 0 \iff \underbrace{\begin{pmatrix}m-2 & m-2 & m-2\\1 & m-2 & 1\\2 & 3 & m-2\end{pmatrix}}_{=M}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

a ovaj sistem će imati netrivijalna rješenja za $\det M \neq 0$

$$\det M = \begin{vmatrix}m-2 & m-2 & m-2\\1 & m-2 & 1\\2 & 3 & m-2\end{vmatrix} = (m-2)\begin{vmatrix}1 & 1 & 1\\1 & m-2 & 1\\2 & 3 & m-2\end{vmatrix} = \ldots = (m-2)(m-3)(m-4)$$

Za $m \in \{2,3,4\}$ dati vektori nisu baza prostora $\mathbb{R}^3$.

Za $m=4$ imamo $\vec{a} = \begin{pmatrix}2\\1\\2\end{pmatrix}$, $\vec{b} = \begin{pmatrix}2\\2\\3\end{pmatrix}$, $\vec{c} = \begin{pmatrix}2\\1\\2\end{pmatrix}$

$$\vec{c} = \eta\begin{pmatrix}2\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\3\end{pmatrix} \implies \begin{matrix}\eta = 1\\ \mu = 0\end{matrix} \qquad \vec{c} = \vec{a} + 0\cdot\vec{b}.$$

Ako je $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b}_1 = \vec{a}_2 + 3\vec{a}_3$, $\vec{b}_2 = \vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ i $\vec{b}_3 = 2\vec{a}_1 + 2\vec{a}_2 + 6\vec{a}_3$ također čine bazu prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c} = -\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3$ preko vektora baze $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

Rj.-upute:

Vektori $\vec{b}_1, \vec{b}_2$ i $\vec{b}_3$ će činiti bazu prostora $\mathbb{R}^3$ ako su linearno nezavisni tj. ako je jedino rješenje sistema

$$\alpha \vec{b}_1 + \beta \vec{b}_2 + \gamma \vec{b}_3 = \vec{0}$$

trivijalno rješenje $\alpha = \beta = \gamma = 0$. Posmatrajmo dati sistem

$$\alpha(\vec{a}_2 + 3\vec{a}_3) + \beta(\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3) + \gamma(2\vec{a}_1 + 2\vec{a}_2 + 6\vec{a}_3) = \vec{0}$$

$$(0 + \beta + 2\gamma)\vec{a}_1 + (\alpha + \beta + 2\gamma)\vec{a}_2 + (3\alpha + 2\beta + 6\gamma)\vec{a}_3 = \vec{0}$$

$$\begin{aligned} \beta + 2\gamma &= 0 \\ \alpha + \beta + 2\gamma &= 0 \\ 3\alpha + 2\beta + 6\gamma &= 0 \end{aligned} \iff \underbrace{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{pmatrix}}_{=M} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det M = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{vmatrix} = \ldots = -2 \neq 0 \Rightarrow$$ vektori $\vec{b}_1, \vec{b}_2$ i $\vec{b}_3$ su linearno nezavisni i oni čine bazu prostora $\mathbb{R}^3$

Odredimo još konstante c_1, c_2 i c_3 t.d. $\vec{c} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$

$$-\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3 = c_1(\vec{a}_2 + 3\vec{a}_3) + c_2(\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3) + c_3(2\vec{a}_1 + 2\vec{a}_2 + 6\vec{a}_3)$$

$$\begin{aligned} c_2 + 2c_3 &= -1 \\ c_1 + c_2 + 2c_3 &= 1 \\ 3c_1 + 2c_2 + 6c_3 &= 2 \end{aligned} \iff \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \\ c_3 &= -1 \end{aligned}$$

Za koje vrijednosti parametra m vektori $\vec{a} = (2m, 1+m, 1)^T$, $\vec{b} = (-m, 1, m)^T$, $\vec{c} = (m, 1, m-2)^T$ čine bazu trodimenzionalnog vektorskog prostora?

Rj. Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ će činiti bazu trodimenzionalnog vektorskog prostora akko su linearno nezavisni tj. akko jedino rješenje sistema po nepoznatim α, β i γ

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$$

je trivijalno rješenje $\alpha = \beta = \gamma = 0$. Drugim riječima akko je determinanta $\begin{vmatrix} 2m & -m & m \\ 1+m & 1 & 1 \\ 1 & m & m-2 \end{vmatrix}$ različita od nule.

Pa izračunajmo vrijednost ove determinante.

$$\begin{vmatrix} 2m & -m & m \\ 1+m & 1 & 1 \\ 1 & m & m-2 \end{vmatrix} = m \begin{vmatrix} 2 & -1 & 1 \\ 1+m & 1 & 1 \\ 1 & m & m-2 \end{vmatrix} \xrightarrow[III_k + II_k]{I_k + II_k \cdot 2} m \begin{vmatrix} 0 & -1 & 0 \\ 3+m & 1 & 2 \\ 2m+1 & m & 2m-2 \end{vmatrix} =$$

$$= m \begin{vmatrix} 3+m & 2 \\ 2m+1 & 2m-2 \end{vmatrix} \xrightarrow{II_v + I_v} m \begin{vmatrix} 3+m & 2 \\ 3m+4 & 2m \end{vmatrix} = m(6m + 2m^2 - 6m - 8)$$

$$= m(2m^2 - 8) = 2m(m-2)(m+2)$$

Za $m \neq 0$, $m \neq 2$, $m \neq -2$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ čine bazu trodimenzionalnog vektorskog prostora.

8. Za koju vrijednost parametra p su vektori $\vec{a}_1=(p,-p^2,3)$, $\vec{a}_2=(p-2,1,1)$ i $\vec{a}_3=(-1,p^2+1,-1)$ linearno zavisni? Za najveću dobijenu vrijednost parametra p napisati vektor $\vec{a}_3$ kao linearnu kombinaciju vektora $\vec{a}_1$ i $\vec{a}_2$.

Rj. $\alpha\vec{a}_1+\beta\vec{a}_2+\gamma\vec{a}_3=\vec{0}$

$$M=\begin{bmatrix} p & p-2 & -1 \\ -p^2 & 1 & p^2+1 \\ 3 & 1 & -1 \end{bmatrix},\quad \det M=\begin{vmatrix} p & p-2 & -1 \\ -p^2 & 1 & p^2+1 \\ 3 & 1 & -1 \end{vmatrix} \frac{II_k+III_k}{I_k+III_k\cdot 3} \begin{vmatrix} p-3 & p-3 & -1 \\ 2p^2+3 & p^2+2 & p^2+1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$=(-1)\begin{vmatrix} p-3 & p-3 \\ 2p^2+3 & p^2+2 \end{vmatrix}=(-1)(p-3)\begin{vmatrix} 1 & 1 \\ 2p^2+3 & p^2+2 \end{vmatrix}=$$

$$=(p-3)(p^2+2-2p^2-3)\cdot(-1)=(-1)(p-3)(-p^2-1)=(p-3)(p^2+1)$$

Za $p=3$ vektori $\vec{a}_1$, $\vec{a}_2$ i $\vec{a}_3$ su linearno zavisni.

$\vec{a}_1=(3,-9,3)$, $\vec{a}_2=(1,1,1)$, $\vec{a}_3=(-1,10,-1)$

$\vec{a}_3=\lambda\vec{a}_1+\omega\vec{a}_2$

$(-1,10,-1)=\lambda(3,-9,3)+\omega(1,1,1)$

$$\Rightarrow \begin{array}{r} 3\lambda+\omega=-1 \\ -\;\; -9\lambda+\omega=10 \\ \hline 12\lambda=-11 \\ \lambda=-\frac{11}{12} \end{array} \qquad \begin{array}{l} 3\lambda+\omega=-1 \\ \omega=-1+\frac{33}{12} \\ \omega=\frac{21}{12} \end{array}$$

$\vec{a}_3=-\frac{11}{12}\vec{a}_1+\frac{21}{12}\vec{a}_2$

9. Dati su vektori $\vec{a}=(-1,-3,1)$, $\vec{b}=(\lambda,3,4)$ i $\vec{c}=(-5,-9,1)$. Odrediti parametar λ tako da vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ budu linearno zavisni, pa izraziti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

10. Dati su vektori $\vec{a}=(m^2+1,m,-2)$, $\vec{b}=(m^2,2,-m)$, $\vec{c}=(-2m-1,0,m+2)$. Odrediti sve vrijednosti parametra m tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra m napisati vektor $\vec{a}$ kao linearnu kombinaciju vektora $\vec{b}$ i $\vec{c}$.

Rj. 9. $\lambda=6$, $\vec{a}=\frac{2}{13}\vec{b}+\frac{5}{13}\vec{c}$

10. $m\in\{-2,0,1,3\}$, $m=3$: $\vec{a}=\frac{3}{2}\vec{b}+\frac{1}{2}\vec{c}$

4. Dokazati da vektori $\vec{a}=(1,2,3)$, $\vec{b}=(1,1,1)$ i $\vec{c}=(1,1,2)$ čine bazu vektorskog prostora E^3, pa naći koordinate vektora $\vec{x}=(6,9,14)$ u odnosu na tu bazu.

Rj. Provjerimo da li su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ linearno zavisni.

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$$M=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix},\quad \det M=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} \frac{II_v-I_v}{III_v-I_v} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix}=(-1)\cdot\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=-1\neq 0$$

Sistem ima samo trivijalno rješenje, vektori su linearno nezavisni. Vektori čine bazu.

$\vec{x}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$

$(6,9,14)=\alpha(1,2,3)+\beta(1,1,1)+\gamma(1,1,2)$

$$\begin{array}{lr} \alpha+\beta+\gamma=6 & (1) \\ 2\alpha+\beta+\gamma=9 & (2) \\ 3\alpha+\beta+2\gamma=14 & (3) \end{array} \qquad \begin{array}{ll} (1)-(2): & -\alpha=-3 \\ (3)-(2): & \alpha+\gamma=5 \\ \hline & \alpha=3,\ \gamma=2 \end{array} \qquad \begin{array}{l} \alpha+\beta+\gamma=6 \\ 3+\beta+2=6 \\ \beta=1 \end{array}$$

$\vec{x}=(3,1,2)$ su koordinate vektora $\vec{x}$ u odnosu na bazu E^3.

5. Za koju vrijednost parametra m vektori $\vec{a}=(m,1+m,1-m)$, $\vec{b}=(2m,1-m,1)$ i $\vec{c}=(-2m,m,2m+2)$ čine bazu trodimenzionalnog vektorskog prostora?

Rj. Provjerimo da li su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ linearno zavisni.

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$$M=\begin{bmatrix} m & 2m & -2m \\ 1+m & 1-m & m \\ 1-m & 1 & 2m+2 \end{bmatrix},\quad \det M=\begin{vmatrix} m & 2m & -2m \\ 1+m & 1-m & m \\ 1-m & 1 & 2m+2 \end{vmatrix} \frac{II_k+III_k}{III_k+I_k\cdot 2}$$

$$=\begin{vmatrix} m & 0 & 0 \\ 1+m & 1 & 3m+2 \\ 1-m & 2m+3 & 4 \end{vmatrix}=m\begin{vmatrix} 1 & 3m+2 \\ 2m+3 & 4 \end{vmatrix}=m\big(4-\overbrace{(3m+2)(2m+3)}^{6m^2+13m+6}\big)=$$

$$=m(4-6m^2-13m-6)=m(-6m^2-13m-2)=m\cdot(-6)(m+2)\left(m+\frac{1}{6}\right)$$

$D=169-48=121$, $x_{1,2}=\frac{13\pm 11}{-12}$, $x_1=-2$, $x_2=-\frac{2}{12}=-\frac{1}{6}$

Za $m\neq 0$, $m\neq -2$ i $m\neq -\frac{1}{6}$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ čine bazu trodimenzionalnog vektorskog prostora.

6. Ako je $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ jedna baza vektorskog prostora V_3, dokazati da i vektori $\vec{b}_1 = \vec{a}_1 + 3\vec{a}_2$, $\vec{b}_2 = -5\vec{a}_1 + \vec{a}_2 + 4\vec{a}_3$ i $\vec{b}_3 = 2\vec{a}_1 + 2\vec{a}_2 + 6\vec{a}_3$ takođe čine bazu prostora V_3 i izraziti vektor $\vec{c} = 11\vec{a}_1 + 3\vec{a}_2 + 14\vec{a}_3$ preko vektora baze $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

Rj. $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ baza vektorskog prostora

$\vec{b}_1 = (1, 0, 3)$, $\vec{b}_2 = (-5, 1, 4)$, $\vec{b}_3 = (2, 2, 6)$ koordinate vektora u $\vec{b}_1, \vec{b}_2, \vec{b}_3$ odnosu na bazu $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

Provjerimo da li su vektori $\vec{b}_1$, $\vec{b}_2$ i $\vec{b}_2$ linearno zavisni:

$\alpha\vec{b}_1 + \beta\vec{b}_2 + \gamma\vec{b}_3 = \vec{0}$

$$M = \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 2 \\ 3 & 4 & 6 \end{bmatrix}, \quad \det M = \begin{vmatrix} 1 & -5 & 2 \\ 0 & 1 & 2 \\ 3 & 4 & 6 \end{vmatrix} \xlongequal{III_k - II_k \cdot 2} \begin{vmatrix} 1 & -5 & 12 \\ 0 & 1 & 0 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 12 \\ 3 & -2 \end{vmatrix} = -38$$

$\det M \neq 0$. Vektori $\vec{b}_1, \vec{b}_2$ i $\vec{b}_3$ su linearno nezavisni, pa oni takođe čine bazu prostora V_3.

$\vec{c} = (11, 3, 14)$ koordinate vektora $\vec{c}$ u odnosu na bazu $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

$\vec{c} = \alpha\vec{b}_1 + \beta\vec{b}_2 + \gamma\vec{b}_3$

$(11, 3, 14) = \alpha(1, 0, 3) + \beta(-5, 1, 4) + \gamma(2, 2, 6)$

$$\begin{aligned} \alpha - 5\beta + 2\gamma &= 11 \quad (1) \\ \beta + 2\gamma &= 3 \quad (2) \\ 3\alpha + 4\beta + 6\gamma &= 14 \quad (3) \end{aligned}$$

$(1)-(2)$: $\alpha - 6\beta = 8$ (I)

$(3)-(2)\cdot 3$: $3\alpha + \beta = 5$ (II)

$(I)+6\cdot(II)$: $19\alpha = 38$, $\alpha = 2$

$3\alpha + \beta = 5$, $6 + \beta = 5$, $\beta = -1$

$\beta + 2\gamma = 3$, $-1 + 2\gamma = 3$, $\gamma = 2$

$\vec{c} = 2\vec{b}_1 - \vec{b}_2 + 2\vec{b}_3 = (2, -1, 2)$ vektor $\vec{c}$ izražen preko vektora baze $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

7. Date su dvije baze vektorskog prostora E^3 $B_1 = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ i $B_2 = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ gdje su $\vec{a}_1 = (1, 1, 2)$, $\vec{a}_2 = (2, 3, -1)$, $\vec{a}_3 = (-1, 0, 1)$ i $\vec{b}_1 = (1, 1, 2)$, $\vec{b}_2 = (2, 1, 0)$ i $\vec{b}_3 = (1, 0, -1)$. Dat je vektor $\vec{x}$ u odnosu na bazu B_1 $\vec{x} = (2, 3, -4)$ odnosno $\vec{x} = 2\vec{a}_1 + 3\vec{a}_2 - 4\vec{a}_3$. Odrediti koordinate vektora $\vec{x}$ u odnosu na bazu B_2.

Rj. $\vec{x} = (3, 8, -7)$

(#) Dati su vektori $\vec{a} = (3m+3, 1, m+5)$, $\vec{b} = (3m-4, 3m-2, -2)$, $\vec{c} = (3-3m, 2-3m, 1)$. Odrediti sve vrijednosti parametra m tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra m napisati vektor $\vec{a}$ kao linearnu kombinaciju vektora $\vec{b}$ i $\vec{c}$.

Rj. Vektori $\vec{a}, \vec{b}$ i $\vec{c}$ su linearno zavisni ako postoje skalari α, β, γ, bar jedan različit od nule, takvi da

$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$.

$\alpha(3m+3, 1, m+5) + \beta(3m-4, 3m-2, -2) + \gamma(3-3m, 2-3m, 1) = 0$

$$\begin{aligned} (3m+3)\alpha + (3m-4)\beta + (3-3m)\gamma &= 0 \\ \alpha + (3m-2)\beta + (2-3m)\gamma &= 0 \\ (m+5)\alpha - 2\beta + \gamma &= 0 \end{aligned}$$

Ovaj (homogeni) sistem ima netrivijalna rješenja akko je $D = 0$.

$$D = \begin{vmatrix} 3m+3 & 3m-4 & 3-3m \\ 1 & 3m-2 & 2-3m \\ m+5 & -2 & 1 \end{vmatrix} \xlongequal{II_k + III_k} \begin{vmatrix} 3m+3 & -1 & 3-3m \\ 1 & 0 & 2-3m \\ m+5 & -1 & 1 \end{vmatrix}$$

$$\xlongequal{I_v - III_v} \begin{vmatrix} 2m-2 & 0 & 2-3m \\ 1 & 0 & 2-3m \\ m+5 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2m-2 & 2-3m \\ 1 & 2-3m \end{vmatrix} \xlongequal{I_v - II_v} \begin{vmatrix} 2m-3 & 0 \\ 1 & 2-3m \end{vmatrix}$$

$= (2m-3)(2-3m)$ $\quad D = 0$ akko $m = \frac{3}{2}$ ili $m = \frac{2}{3}$

$\frac{3}{2} > \frac{2}{3} \Rightarrow m = \frac{3}{2}$: $\vec{a} = \left(\frac{9}{2} + 3, 1, \frac{3}{2} + 5\right) = \left(\frac{15}{2}, 1, \frac{13}{2}\right)$

$\vec{b} = \left(\frac{9}{2} - 4, \frac{9}{2} - 2, -2\right) = \left(\frac{1}{2}, \frac{5}{2}, -2\right)$ $\quad \vec{c} = \left(3 - \frac{9}{2}, 2 - \frac{9}{2}, 1\right) = \left(-\frac{3}{2}, -\frac{5}{2}, 1\right)$

$\vec{a} = \mu\vec{b} + \eta\vec{c}$ – razlaganje vektora $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$

Pronađimo vrijednosti μ i η.

$\left(\frac{15}{2}, 1, \frac{13}{2}\right) = \mu\left(\frac{1}{2}, \frac{5}{2}, -2\right) + \eta\left(-\frac{3}{2}, -\frac{5}{2}, 1\right)$

$$\begin{aligned} \frac{1}{2}\mu - \frac{3}{2}\eta &= \frac{15}{2} \\ \frac{5}{2}\mu - \frac{5}{2}\eta &= 1 \\ -2\mu + \eta &= \frac{13}{2} \end{aligned}$$

$\Rightarrow \mu = -\frac{69}{10}$, $\eta = -\frac{73}{10}$ $\quad \vec{a} = \frac{-69\vec{b} - 73\vec{c}}{10}$ ... sistem riješiti za vježbu

(#) Dati su vektori $\vec{a}=(m^2+1, m, -2)$, $\vec{b}=(m^2, 2, -m)$, $\vec{c}=(-2m-1, 0, m+2)$. Odrediti sve vrijednosti parametra m tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra m napisati vektor $\vec{a}$ kao linearnu kombinaciju vektora $\vec{b}$ i $\vec{c}$.

R: Vektori $\vec{a}, \vec{b}$ i $\vec{c}$ su linearno zavisni ako postoji bar jedan nenula skalar α, β ili γ takav da je

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$ tj.

$$\begin{aligned}(m^2+1)\alpha + m^2\beta + (-2m-1)\gamma &= 0\\ m\alpha + 2\beta + 0\gamma &= 0\\ -2\alpha + (-m)\beta + (m+2)\gamma &= 0\end{aligned}$$

Ovo je homogeni sistem.

Za $D=0$ sistem ima netrivijalnih rješenja.

$$D=\begin{vmatrix} m^2+1 & m^2 & -2m-1\\ m & 2 & 0\\ -2 & -m & m+2\end{vmatrix} = -m\begin{vmatrix} m^2 & -2m-1\\ -m & m+2\end{vmatrix} + 2\begin{vmatrix} m^2+1 & -2m-1\\ -2 & m+2\end{vmatrix} =$$

$$= -m\left(m^3+2m^2-(2m^2+m)\right)+2\left(m^3+2m^2+m+2-(4m+2)\right) =$$

$$= -m(m^3-m)+2(m^3+2m^2-3m) = -m^2(m^2-1)+2m(m^2+2m-3) =$$

$D=4+12=16$, $m_{1,2}=\frac{-2\pm 4}{2}$, $m_1=\frac{-6}{2}=-3$, $m_2=\frac{2}{2}=1$

$$= m\left[-m(m-1)(m+1)+2(m-1)(m+3)\right] = m(m-1)\left[-m(m+1)+2(m+3)\right] =$$

$$= m(m-1)(-m^2-m+2m+6) = m(m-1)(-m+m+6) = -m(m-1)(m+2)(m-3)$$

$D=0$ akko $m=0$ ili $m=1$ ili $m=-2$ ili $m=3$

Vektori $\vec{a}, \vec{b}$ i $\vec{c}$ su linearno zavisni ako $m\in\{-2,0,1,3\}$

Za $m=3$: $\vec{a}=(10,3,-2)$, $\vec{b}=(9,2,-3)$ i $\vec{c}=(-7,0,5)$

$\vec{a}=\mu\vec{b}+\omega\vec{c}$

$(9\mu, 2\mu, -3\mu)+(-7\omega, 0, 5\omega)=(10,3,-2)$

$$\begin{aligned}9\mu-7\omega&=10\\ 2\mu+0&=3\\ -3\mu+5\omega&=-2\end{aligned}$$

$\mu=\frac{3}{2}$

$-\frac{9}{2}+5\omega=-2 \quad /\cdot 2$

$-9+10\omega=-4$

$10\omega=5$

$\omega=\frac{1}{2}$

$\vec{a}=\frac{3}{2}\vec{b}+\frac{1}{2}\vec{c}$

vektor $\vec{a}$ razložen preko vektora $\vec{b}$ i $\vec{c}$

Razvijmo determinantu D i na drugi način:

$$D=\begin{vmatrix} m^2+1 & m^2 & -2m-1\\ m & 2 & 0\\ -2 & -m & m+2\end{vmatrix} \underset{=}{I_R+III_R} \begin{vmatrix} m^2-1 & m^2-m & -m+1\\ m & 2 & 0\\ -2 & -m & m+2\end{vmatrix} =$$

$$= \begin{vmatrix} (m-1)(m+1) & m(m-1) & -(m-1)\\ m & 2 & 0\\ -2 & -m & m+2\end{vmatrix} = (m-1)\begin{vmatrix} m+1 & m & -1\\ m & 2 & 0\\ -2 & -m & m+2\end{vmatrix} \underset{=}{I_k+III_k}$$

$$=(m-1)\begin{vmatrix} m & m & -1\\ m & 2 & 0\\ m & -m & m+2\end{vmatrix} = m(m-1)\begin{vmatrix} 1 & m & -1\\ 1 & 2 & 0\\ 1 & -m & m+2\end{vmatrix} \underset{III_R-II_R}{\overset{I_R-II_R}{=}}$$

$$= m(m-1)\begin{vmatrix} 0 & m-2 & -1\\ 1 & 2 & 0\\ 0 & -m-2 & m+2\end{vmatrix} = -m(m-1)\begin{vmatrix} m-2 & -1\\ -(m+2) & m+2\end{vmatrix} = -m(m-1)(m+2)\begin{vmatrix} m-2 & -1\\ -1 & 1\end{vmatrix}$$

$$= -m(m-1)(m+2)(m-2-1) = -m(m-1)(m+2)(m-3)$$

Brojni nizovi

Brojni niz je realna f-ja definisana nad skupom prirodnih brojeva.

Npr.

$1, 2, 3, \ldots, n, n+1, \ldots$ je niz prirodnih brojeva. Opšti član ovoy niza je $a_n = n$, $n \in \mathbb{N}$. Niz možemo pisati i u obliku $\{n\}_{n\in\mathbb{N}}$.

$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots$ je niz sa opštim članom $b_n = \frac{1}{n}$, $n \in \mathbb{N}$. Ovaj niz možemo pisati i u obliku $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \ldots$ je niz čiji je opšti član $s_n = \frac{(-1)^n}{n^2}$, $n\in\mathbb{N}$. Skraćeno niz možemo pisati kao $\left\{\frac{(-1)^n}{n^2}\right\}_{n\in\mathbb{N}}$

$\frac{1}{2}, -1, \frac{3}{2}, -2, \frac{5}{2}, -3, \ldots$ je niz čiji je opšti član $t_n = \frac{(-1)^{n-1} \cdot n}{2}$. Niz možemo pisati u obliku $\left\{\frac{(-1)^{n-1}}{2} \cdot n\right\}_{n\in\mathbb{N}}$

Aritmetički niz

Aritmetički niz je niz brojeva kod kojih je razlika između dva susjedna člana stalan broj.

$a_1, a_2, a_3, a_4, \ldots, a_n, a_{n+1}, \ldots$

$a_2 - a_1 = d$
$a_3 - a_2 = d$
$a_4 - a_3 = d$
$\vdots$
$a_n - a_{n-1} = d$
$\vdots$

a_1
$a_2 = a_1 + d$
$a_3 = a_2 + d = a_1 + 2d$
$a_4 = a_3 + d = a_1 + 3d$
$\vdots$
$a_n = a_{n-1} + d = a_1 + (n-1)d$
$\vdots$

$s+t = n+1$
$a_s + a_t = a_1 + (s-1)d + a_1 + (t-1)d = 2a_1 + (s+t-2)d = 2a_1 + (n-1)d = a_1 + a_n$

$$
\begin{aligned}
S_n &= a_1 + a_2 + \ldots + a_n \\
+\ S_n &= a_n + a_{n-1} + \ldots + a_1 \\
\hline
2S_n &= (a_1+a_n) + (a_2 + a_{n-1}) + \ldots + (a_n + a_1)
\end{aligned}
$$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d)$$

suma prvih n članova

1. Izračunati sumu prvih 20 članova niza $2, 5, 8, 11, 14, \ldots$

Rj. Ovo je aritmetički niz, $d = 3$

$a_{20} = a_{19} + 3 = a_1 + 19 \cdot 3 = 2 + 57 = 59$

$S_n = \frac{n}{2}(a_1 + a_n) = \frac{20}{2}(2 + 59) = 10 \cdot 61 = 610$ suma prvih dvadeset članova

Geometriski niz

Geometriski niz je niz brojeva kod kojeg je količnik dva susjedna člana stalan broj.

$b_1, b_2, b_3, b_4, \ldots b_{n-1}, b_n, \ldots$

$b_2 : b_1 = q$
$b_3 : b_2 = q$
$b_4 : b_3 = q$
$\vdots$
$b_n : b_{n-1} = q$
$\vdots$

b_1
$b_2 = b_1 q$
$b_3 = b_2 q = b_1 q^2$
$b_4 = b_3 q = b_1 q^3$
$\vdots$
$b_n = b_{n-1} q = b_1 q^{n-1}$
$\vdots$

$$
\begin{aligned}
S_n &= b_1 + b_2 + b_3 + \ldots + b_n \\
S_n &= b_1 + b_1 q + b_1 q^2 + \ldots + b_1 q^{n-1} \\
S_n &= b_1(1 + q + q^2 + \ldots + q^{n-1}) \quad / (1-q) \\
(1-q)S_n &= b_1(1-q)(1 + q + q^2 + \ldots + q^{n-1}) \\
(1-q)S_n &= b_1(1-q^n) \quad / : (1-q) \\
S_n &= b_1 \frac{1-q^n}{1-q}
\end{aligned}
$$

suma prvih n članova

2. Izračunati sumu prvih 50 članova niza $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

Rj. Ovo je geometriski niz. $b_1 = \frac{1}{3}$, $q = \frac{1}{3}$, $S_n = b_1 \frac{1-q^n}{1-q}$.

$$S_{50} = \frac{1}{3} \cdot \frac{1 - \left(\frac{1}{3}\right)^{50}}{1 - \frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} \cdot \left(1 - \frac{1}{3^{50}}\right) = \frac{1}{2}\left(1 - \frac{1}{3^{50}}\right) = \frac{1}{2} - \frac{1}{2 \cdot 3^{50}} \approx \frac{1}{2}$$

Monotoni nizovi

Ako je $x_n < x_{n+1}$ tada niz $\{x_n\}_{n\in\mathbb{N}}$ raste

$x_n \le x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne opada

$x_n > x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ opada

$x_n \ge x_{n+1} \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ ne raste

ove nizove jednim imenom zovemo monotoni nizovi

$$a_{n+1} - a_n = \ldots \begin{cases} < 0, & \text{niz opada} \\ > 0, & \text{niz raste} \end{cases} \qquad \frac{a_{n+1}}{a_n} = \ldots \begin{cases} > 1, & \text{rastući niz} \\ < 1, & \text{opadajući niz} \end{cases}$$

3. Ispitati monotonost niza $\{a_n\}_{n\in\mathbb{N}}$ gdje je $a_n = \frac{n-1}{2n+1}$.

Rj.
$$a_{n+1} - a_n = \frac{n+1-1}{2(n+1)+1} - \frac{n-1}{2n+1} = \frac{n}{2n+3} - \frac{n-1}{2n+1} = \frac{2n^2 + n - (2n^2 - 2n + 3n - 3)}{(2n+3)(2n+1)}$$
$$= \frac{3}{(2n+3)(2n+1)} > 0, \ \forall n \quad \Rightarrow \{a_n\} \text{ je rastući niz}$$

Granična vrijednost niza

Broj A nazivamo granična vrijednost niza ili limesom niza realnih brojeva $x_1, x_2, \dots, x_n, \dots$, što simbolički pišemo

$$\lim_{n\to\infty} x_n = A$$

ako za svaki $\varepsilon > 0$ postoji broj N (koji zavisi od ε) tako da

$$|x_n - A| < \varepsilon \quad \text{za svaki } n > N.$$

1. Dat je niz $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$ Izračunati za koju vrijednost n će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$ ako je $\varepsilon = 0{,}001$.

Rj. $\frac{1}{n^2} < 0{,}001$

$\frac{1}{n^2} < 10^{-3} \quad / \cdot n^2$

$10^{-3} n^2 > 1 \quad / \cdot 10^3$

$n^2 > 10^3$

$n > 10\sqrt{10} \approx 31{,}62$

Za sve $n > 31$ će biti zadovoljena nejednakost $\frac{1}{n^2} < \varepsilon$.

2. Pokazati da je $\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$.

Rj. Iz definicije $\forall \varepsilon > 0 \; \exists N$ (koji zavisi od ε) tako da

$$\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon \quad \text{za svaki } n > N.$$

$$\left| \frac{2n+1}{n+1} - 2 \right| = \left| \frac{2n+1-2n-2}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \varepsilon$$

$(n+1)\varepsilon > 1 \quad / : \varepsilon \; (\varepsilon > 0)$

$n+1 > \frac{1}{\varepsilon}$

$n > \frac{1}{\varepsilon} - 1$

Prema tome za svaki pozitivan broj ε postoji takav broj N $\left(N = \frac{1}{\varepsilon} - 1\right)$ takav da za $n > N$ vrijedi $\left| \frac{2n+1}{n+1} - 2 \right| < \varepsilon$.

Prema tome $\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$.

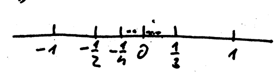
3. Odredite limese nizova

a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, \frac{(-1)^{n-1}}{n}, \dots$;

b) $\frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \dots, \frac{2n}{2n-1}, \dots$;

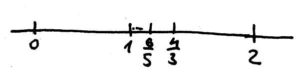
c) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$;

Rj. a)



0

b)



1

c) $\sqrt{2} \approx 1{,}41$

$\sqrt{2\sqrt{2}} \approx \sqrt[4]{8} \approx 1{,}68$

$\sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2\sqrt[4]{8}} = \sqrt[8]{2^7} \approx 1{,}83$

Dat je niz $v_1 = \frac{\cos\frac{\pi}{2}}{1}$, $v_2 = \frac{\cos\pi}{2}$, $v_3 = \frac{\cos\frac{3\pi}{2}}{3}, \dots$, $v_n = \frac{\cos\frac{n\pi}{2}}{n}$. Nađi $\lim\limits_{n\to\infty} v_n$. Koliko mora biti n da bi apsolutna vrijednost razlike između v_n i $\lim\limits_{n\to\infty} v_n$ bila ne veća od $0{,}0001$?

Rj. $\cos\frac{\pi}{2} = 0$, $\cos\pi = -1$, $\cos\frac{3\pi}{2} = 0$, $\cos 2\pi = 1, \dots$

Kolko iznosi $\cos\frac{n\pi}{2}$.

Za n neparno tj. za n oblika $n = 2k+1$, $k \in \mathbb{N}$

$$\cos\frac{(2k+1)\pi}{2} = 0$$

Za n parno tj. za n oblika $n = 2k$, $k \in \mathbb{N}$

$$\cos\frac{2k\pi}{2} = (-1)^{k+1}$$

Napišimo sad neke članove niza (v_n)

$$0, \frac{-1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{6}, 0, \frac{1}{8}, 0, \frac{-1}{10}, 0, \dots \qquad \dots (*)$$

Prema definiciji $\lim\limits_{n\to\infty} v_n = A$ ako i samo ako $\forall \varepsilon > 0 \; \exists$ prirodan broj N (N zavisi samo od ε) ~~takav~~ tako da

$$|v_n - A| < \varepsilon \quad \text{za svaki } n > N.$$

$$\left| \frac{\cos\frac{n\pi}{2}}{n} - A \right| < \varepsilon \iff -\varepsilon < \frac{\cos\frac{n\pi}{2}}{n} - A < \varepsilon$$

$$A - \varepsilon < \frac{\cos\frac{n\pi}{2}}{n} < A + \varepsilon$$

Iz ove definicije ili iz $(*)$ možemo zaključiti da je $A = 0$ tj. $\lim\limits_{n\to\infty} v_n = 0$. Uzmimo da je $\varepsilon = 0{,}0001 = 10^{-4}$.

$\left| \frac{\cos\frac{n\pi}{2}}{n} - 0 \right| < 10^{-4}$ Kako je $\left|\cos\frac{n\pi}{2}\right| < 1 \; \forall n$ imamo

$\frac{1}{n} < 10^{-4} \Rightarrow n > \frac{1}{10^{-4}} = 10^4$. Za svako $n > 10^4$ apsolutna vrijednost razlike između v_n i $\lim\limits_{n\to\infty} v_n$ je manja od $0{,}0001$.

Naći vrijednost sljedećih f-ja

a) $f(x)=2x-3-\frac{1}{x}$ kada $x\to 1$;

b) $f(x)=\frac{x^3-3x^2+2x-5}{x^2+2}$ kada $x\to -1$

c) $y = x\sin\frac{1}{x}$ kada $x\to 0$.

Rj.

a) $\lim\limits_{x\to 1}\left(2x-3-\frac{1}{x}\right) = 2\cdot 1-3-1 = -2$;

b) $\lim\limits_{x\to -1}\frac{x^3-3x^2+2x-5}{x^2+2} = \frac{(-1)^3-3(-1)^2+2\cdot(-1)-5}{(-1)^2+2} =$

$= \frac{-1-3-2-5}{1+2} = \frac{-11}{3}$

c) Primjetimo da $\lim\limits_{x\to 0}\frac{1}{x} = \infty$

Pa je $\lim\limits_{x\to 0}\sin\frac{1}{x}$ $\underbrace{\text{neodređen izraz}}_{}$ (ali uvijek je između -1 i 1).

Međutim kako je $|\sin\frac{1}{x}|\le 1$ za svako x i nula pomnožena sa bilo kojim konačnim brojem je nula to

$\lim\limits_{x\to 0} x\cdot\sin\frac{1}{x} = 0$.

Napomena: $0\cdot\infty$ je neodređen izraz

$\sqrt{2},\ \sqrt[4]{2^3},\ \sqrt[8]{2^7},\ \ldots,\ \sqrt[2^n]{2^{2^n-1}},\qquad \lim\limits_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = 1$

Operacije sa limesima

a) $\lim\limits_{n\to\infty}(a_n\pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

b) $\lim\limits_{n\to\infty}(a_n\cdot b_n) = \lim\limits_{n\to\infty} a_n\cdot\lim\limits_{n\to\infty} b_n$

c) $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$

d) $\lim\limits_{n\to\infty}\sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n}$

e) $\lim\limits_{n\to\infty} b^{a_n} = b^{\lim\limits_{n\to\infty} a_n}$, $b>0$

f) $\lim\limits_{n\to\infty}\log_b a_n = \log_b\lim\limits_{n\to\infty} a_n$, $b>1$

1. Izračunajte limese

a) $\lim\limits_{n\to\infty}\frac{1}{n}$ Rj. $\lim\limits_{n\to\infty}\frac{1}{n} = 0$

b) $\lim\limits_{n\to\infty} 7$ Rj. $\lim\limits_{n\to\infty} 7 = 7$

c) $\lim\limits_{n\to\infty} n^2$ Rj. $\lim\limits_{n\to\infty} n^2 = \infty$

d) $\lim\limits_{n\to\infty}\frac{n}{n+1}$

Rj. $\lim\limits_{n\to\infty}\frac{n}{n+1}\left(=\frac{\infty}{\infty}\right)\begin{matrix}/:n\\/:n\end{matrix} = \lim\limits_{n\to\infty}\frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}} = 1$

e) $\lim\limits_{n\to\infty}\frac{n^2+n-3}{n^3+n^2+1}\begin{matrix}/:n^3\\/:n^3\end{matrix}$ Rj. 0

Neodređeni izrazi su $\frac{0}{0}$, $\infty-\infty$, $0\cdot\infty$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$

Određeni izrazi su $\infty\cdot\infty=\infty$, $\infty+\infty=\infty$, $\frac{0}{\infty}=0$

2. Izračunati limese:

a) $\lim\limits_{n\to\infty}\frac{n^3+3n+9}{2n^2+3n-1}$ Rj. $\lim\limits_{n\to\infty}\frac{n^3+3n+9}{2n^2+3n-1}\begin{matrix}/:n^3\\/:n^3\end{matrix} = \lim\limits_{n\to\infty}\frac{1+\underbrace{\frac{3}{n^2}}_{\to 0}+\underbrace{\frac{9}{n^3}}_{\to 0}}{\underbrace{\frac{2}{n}}_{\to 0}+\underbrace{\frac{3}{n^2}}_{\to 0}-\underbrace{\frac{1}{n^3}}_{\to 0}} = \frac{1}{0} = \infty$

b) $\lim\limits_{n\to\infty}\frac{n^2+2n+3}{2n^2+n-4}$ Rj. $\lim\limits_{n\to\infty}\frac{n^2+2n+3}{2n^2+n-4}\begin{matrix}/:n^2\\/:n^2\end{matrix} = \lim\limits_{n\to\infty}\frac{1+\frac{2}{n}+\frac{3}{n^2}}{2+\frac{1}{n}-\frac{4}{n^2}} = \frac{1}{2}$

c) $\lim\limits_{n\to\infty}\frac{3n^3+n-1}{2n^4+1}$ Rj. $\lim\limits_{n\to\infty}\frac{3n^3+n-1}{2n^4+1}\begin{matrix}/:n^4\\/:n^4\end{matrix} = \lim\limits_{n\to\infty}\frac{\frac{3}{n}+\frac{1}{n^3}-\frac{1}{n^4}}{2+\frac{1}{n^4}} = \frac{0}{2} = 0$

d) $\lim\limits_{n\to\infty}\frac{(n+1)(n+2)(n+3)}{n^3}$ Rj. $\lim\limits_{n\to\infty}\frac{(n+1)(n+2)(n+3)}{n^3}\begin{matrix}/:n^3\\/:n^3\end{matrix} = \lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})(1+\frac{2}{n})(1+\frac{3}{n})}{1} = \frac{1}{1} = 1$

e) $\lim\limits_{n\to\infty}\frac{n+(-1)^n}{3n-(-1)^n}$ Rj. $\lim\limits_{n\to\infty}\frac{n+(-1)^n}{3n-(-1)^n}\begin{matrix}/:n\\/:n\end{matrix} = \lim\limits_{n\to\infty}\frac{1+\overbrace{\frac{(-1)^n}{n}}^{\to 0}}{3-\underbrace{\frac{(-1)^n}{n}}_{\to 0}} = \frac{1}{3}$

3. Izračunati limese:

a) ✓ $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)$

b) $\lim\limits_{n\to\infty}\left(\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)$

c) ✓ $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right)$

d) $\lim\limits_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right)$

Rj. a) ✓ rj. $\frac{1}{2}$ c) ✓ Rj. 1

b) $$\lim_{n\to\infty}\left(\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim_{n\to\infty}\left(\frac{\frac{n}{2}(1+2n-1)}{n+1}-\frac{2n+1}{2}\right)=\lim_{n\to\infty}\left(\frac{2n^2}{2n+2}-\frac{2n+1}{2}\right)$$
$$=\lim_{n\to\infty}\frac{2n^2-\overbrace{(2n+1)(n+1)}^{2n^2+3n+1}}{2(n+1)}=\lim_{n\to\infty}\frac{\cancel{2n^2}-\cancel{2n^2}-3n-1}{2n+2}\;\begin{matrix}/:n\\/:n\end{matrix}=\lim_{n\to\infty}\frac{-3-\frac{1}{n}}{2+\frac{2}{n}}=-\frac{3}{2}$$

d) imamo niz $1,-\frac{1}{3},\frac{1}{9},-\frac{1}{27},\ldots$ količnik dva susjedna člana je $-\frac{1}{3}$

imamo geometrijski niz, $|q|<1$, $S_n=a_1\frac{1-q^n}{1-q}$

$$\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{9}-\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right)=\lim_{n\to\infty}\left(1\cdot\frac{1-\overbrace{\left(-\frac{1}{3}\right)^n}^{\to 0}}{1-\left(-\frac{1}{3}\right)}\right)=\frac{1}{1+\frac{1}{3}}=\frac{1}{\frac{4}{3}}=\frac{3}{4}$$

4. Izračunati limese:

a) $\lim\limits_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}$ b) $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ c) ✓ $\lim\limits_{n\to\infty}\frac{n\sin n!}{n^2+1}$

d) ✓ $\lim\limits_{x\to\infty}\frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1}$ e) ✓ $\lim\limits_{x\to\infty}\frac{1000x}{x^2-1}$ f) ✓ $\lim\limits_{x\to\infty}\frac{2x^2-x^3-4}{\sqrt{x^4+1}}$

g) ✓ $\lim\limits_{x\to\infty}\frac{2x+3}{x+\sqrt[3]{x}}$ h) ✓ $\lim\limits_{x\to\infty}\frac{x^2}{10+x\sqrt{x}}$ i) $\lim\limits_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$

Rj. a) $$\lim_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\;\begin{matrix}/:3^n\\/:3^n\end{matrix}=\lim_{n\to\infty}\frac{\overbrace{\frac{2^{n+1}}{3^n}}^{0}+3}{\underbrace{\left(\frac{2}{3}\right)^n}_{0}+1}=3$$

b) $$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\infty}=0$$

i) $$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim_{x\to+\infty}\left(\frac{x}{x+\sqrt{x+\sqrt{x}}}\right)^{\frac{1}{2}}=\lim_{x\to+\infty}\left(\frac{1}{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}\right)^{\frac{1}{2}}=1$$

c) ✓ rj. 0 d) ✓ rj. 8 e) ✓ rj. 0 f) ✓ rj. 2 g) ✓ rj. 2 h) ∞

Granična vrijednost f-je

Kažemo da f-ja $f(x)\to A$ kada $x\to p$ (A i p su brojevi) ili da je $\lim\limits_{x\to a}f(x)=A$ ako za svaki $\varepsilon>0$ postoji takav $\delta>0$ (δ zavisi od ε) da je $|f(x)-A|<\varepsilon$ za $0<|x-p|<\delta$.

1. Izračunati limese:

a) $$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}\left(=\frac{0}{0}\right)=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-1)}=\lim_{x\to 2}\frac{x+2}{x-1}=4$$

b) $$\lim_{x\to -1}\frac{x^3+1}{x^2+1}=\frac{0}{2}=0$$

c) $$\lim_{x\to 5}\frac{x^2-5x+10}{x^2-25}=\frac{25-25+10}{0}=\infty$$

d) $$\lim_{x\to -1}\frac{x^2-1}{x^2+3x+2}\left(=\frac{0}{0}\right)=\lim_{x\to -1}\frac{(x-1)(x+1)}{(x+1)(x+2)}=\lim_{x\to -1}\frac{x-1}{x+2}=\frac{-2}{1}=-2$$

e) $$\lim_{x\to 2}\frac{x^2-2x}{x^2-4x+4}\left(=\frac{0}{0}\right)=\lim_{x\to 2}\frac{x(x-2)}{(x-2)^2}=\lim_{x\to 2}\frac{x}{x-2}=\frac{2}{0}=+\infty$$

f) ✓ $\lim\limits_{x\to 1}\frac{x^3-3x+2}{x^4-4x+3}$ Rj. $\frac{1}{2}$

g) $$\lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3}\left(=\frac{0}{0}\right)=\lim_{x\to a}\frac{(x-a)(x-1)}{(x-a)(x^2+ax+a^2)}=\frac{a-1}{a^2+a^2+a^2}=\frac{a-1}{3a^2}$$

$$\Big(x^2\overbrace{-(a+1)x}^{-ax-x}+a\Big):(x-a)=x-1$$
$$\begin{array}{l}-\;x^2-ax\\ \hline \quad -x+a\\ \quad -x+a\\ \hline \quad = \;\; =\end{array}$$

$$\begin{array}{c}1\\1\;\;1\\1\;\;2\;\;1\\1\;\;3\;\;3\;\;1\end{array}$$

h) $$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}\left(=\frac{0}{0}\right)=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

i) ✓ $\lim\limits_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$ Rj. -1.

2. Izračunati limese

a) $\lim\limits_{x\to0}\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}\ \left(=\frac00\right)=\left|\begin{array}{l}\text{uvodimo smjenu}\\ 1+x=y^6\\ x\to0\Rightarrow y\to1\end{array}\right|=\lim\limits_{y\to1}\frac{y^3-1}{y^2-1}=\lim\limits_{y\to1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}=\frac32$

b) $\lim\limits_{x\to1}\frac{\sqrt{x}-1}{x-1}\ \left(=\frac00\right)=\left|\begin{array}{l}x=t^2\\ x\to1\Rightarrow t\to1\end{array}\right|=\lim\limits_{t\to1}\frac{t-1}{t^2-1}=\lim\limits_{t\to1}\frac{t-1}{(t-1)(t+1)}=\frac12$

c)ˇ $\lim\limits_{x\to64}\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$ Rj. 3

d) $\lim\limits_{x\to1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\ \left(=\frac00\right)=\left|\begin{array}{l}x=t^{12}\\ x\to1\Rightarrow t\to1\end{array}\right|=\lim\limits_{t\to1}\frac{\overbrace{t^4-1}^{(t^2-1)(t^2+1)}}{t^3-1}=\lim\limits_{t\to1}\frac{\cancel{(t-1)}(t+1)(t^2+1)}{\cancel{(t-1)}(t^2+t+1)}=\frac43$

e)ˇ $\lim\limits_{x\to1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ Rj. $\frac19$

3. Izračunati limese

a) $\lim\limits_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}\ \left(=\frac00\right)=\lim\limits_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim\limits_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\frac{1}{2\sqrt{a}}\ (a>0)$

b) $\lim\limits_{x\to7}\frac{2-\sqrt{x-3}}{x^2-49}\ \left(=\frac00\right)=\lim\limits_{x\to7}\frac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{\underbrace{(x^2-49)}_{(x-7)(x+7)}(2+\sqrt{x-3})}=\lim\limits_{x\to7}\frac{\cancel{7-x}}{(-1)\cancel{(7-x)}\underbrace{(x+7)}_{14}\underbrace{(2+\sqrt{x-3})}_{4}}=-\frac{1}{56}$

c)ˇ $\lim\limits_{x\to8}\frac{x-8}{\sqrt[3]{x}-2}$ Rj. 12

d) $\lim\limits_{x\to1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}\ \left(=\frac00\right)=\lim\limits_{x\to1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(\sqrt{x}+1)}=\lim\limits_{x\to1}\frac{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{\cancel{(x-1)}(\sqrt{x}+1)}=\frac32$

e) $\lim\limits_{x\to4}\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\ \left(=\frac00\right)=\lim\limits_{x\to4}\frac{(3-\sqrt{5+x})(3+\sqrt{5+x})(1+\sqrt{5-x})}{(1-\sqrt{5-x})(1+\sqrt{5-x})(3+\sqrt{5+x})}=\lim\limits_{x\to4}\frac{(4-x)(1+\sqrt{5-x})}{\underbrace{(-4+x)}_{(-1)(4-x)}(3+\sqrt{5+x})}=\frac{2}{-6}=-\frac13$

f)ˇ $\lim\limits_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ Rj. 1

g) $\lim\limits_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\ \left(=\frac00\right)=\lim\limits_{h\to0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\lim\limits_{h\to0}\frac{x+h-x}{h\cdot(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$

h)ˇ $\lim\limits_{h\to0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\ (x\neq0)$, Rj. $\frac{1}{3\sqrt[3]{x^2}}$

i)ˇ $\lim\limits_{x\to3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$ Rj. $-\frac13$

4. Izračunati limese

a) $\lim\limits_{x\to+\infty}(\sqrt{x+a}-\sqrt{x})\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\frac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{(\sqrt{x+a}+\sqrt{x})}=\lim\limits_{x\to+\infty}\frac{x+a-x}{(\sqrt{x+a}+\sqrt{x})}=\frac{a}{+\infty}=0$

b) $\lim\limits_{x\to+\infty}[\sqrt{x(x+a)}-x]\ (=\infty-\infty)=\lim\limits_{x\to+\infty}\frac{[\sqrt{x(x+a)}-x][\sqrt{x(x+a)}+x]}{\sqrt{x(x+a)}+x}=\lim\limits_{x\to+\infty}\frac{x^2+ax-x^2}{\sqrt{x(x+a)}+x}=$

$=\lim\limits_{x\to+\infty}\frac{ax}{\sqrt{x(x+a)}+x}\ \begin{array}{l}/:x\\ /:x\end{array}=\lim\limits_{x\to+\infty}\frac{a}{\sqrt{1\cdot(1+\underbrace{\frac{a}{x}}_{\to0})}+1}=\frac{a}{2}$

c)ˇ $\lim\limits_{x\to+\infty}(\sqrt{x^2-5x+6}-x)$ Rj. $-\frac52$

d) $\lim\limits_{x\to+\infty}x(\sqrt{x^2+1}-x)\ (=\infty(\infty-\infty))=\lim\limits_{x\to+\infty}\frac{x(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)}=\lim\limits_{x\to+\infty}\frac{x(x^2+1-x^2)}{(\sqrt{x^2+1}+x)}$

$=\lim\limits_{x\to\infty}\frac{x}{\sqrt{x^2+1}+x}\ \begin{array}{l}/:x\\ /:x\end{array}=\lim\limits_{x\to\infty}\frac{1}{\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to0}}+1}=\frac12$

e)ˇ $\lim\limits_{x\to\infty}(x+\sqrt[3]{1-x^3})$ Rj. 0

Navedimo nekoliko važnih graničnih vrijednosti:

$\lim\limits_{x\to0}\frac{\sin x}{x}=1$ $\qquad$ $\lim\limits_{x\to0}(1+x)^{\frac1x}=e$

$\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)=e^k$ $\qquad$ $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$

$\lim\limits_{n\to\infty}\sqrt[n]{a}=1$ $\qquad$ $\lim\limits_{n\to\infty}\frac{a^n}{n}=\infty$ $\qquad$ $\lim\limits_{n\to\infty}\frac{n^k}{a^n}=0$

5. Izračunati limese

a) $\lim\limits_{x\to0}\frac{\sin5x}{x}=\lim\limits_{x\to0}\left(\frac{\sin5x}{5x}\cdot5\right)=1\cdot5=5$

b) $\lim\limits_{x\to2}\frac{\sin x}{x}=\frac12\sin2$

c) $\lim\limits_{x\to\infty}\frac{\sin x}{x}=\left|\begin{array}{l}\text{kako je}\\ -1\le\sin\alpha\le1\\ \text{za }\forall\alpha\end{array}\right|=0$

d)ˇ $\lim\limits_{x\to0}\frac{\sin3x}{x}$ Rj. 3

e) $\lim\limits_{x\to0}\frac{\sin5x}{\sin2x}=\lim\limits_{x\to0}\frac{\frac{\sin5x}{5x}\cdot5}{\frac{\sin2x}{2x}\cdot2}=\frac52$

e) $\lim_{x\to\pi}\frac{\sin mx}{\sin nx} = \left|\begin{matrix} x=\pi+t \\ x\to\pi \Rightarrow t\to 0\end{matrix}\right| = \lim_{t\to0}\frac{\sin(m\pi+mt)}{\sin(n\pi+nt)} = \lim_{t\to0}\frac{\overbrace{\sin m\pi}^{=0}\cos mt+\sin mt\cos m\pi}{\underbrace{\sin n\pi}_{=0}\cos nt+\sin nt\cos n\pi}$

$=\lim_{t\to0}\frac{(-1)^m\sin mt}{(-1)^n\sin nt} = (-1)^{m-n}\lim_{t\to0}\frac{\frac{\sin mt}{mt}\cdot mt}{\frac{\sin nt}{nt}\cdot nt} = (-1)^{m-n}\frac{m}{n}$

f) $\lim_{x\to0}\frac{1-\cos x}{x^2} = \lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to0}\frac{2(\sin\frac{x}{2})^2}{4\cdot(\frac{x}{2})^2} = \lim_{x\to0}\frac12\left(\frac{\sin\frac x2}{\frac x2}\right)^2=\frac12$

$$\left.\begin{matrix} 1=\sin^2x+\cos^2x & 1=\sin^2\frac x2+\cos^2\frac x2 \\ \cos2x=\cos^2x-\sin^2x & \cos x=\cos^2\frac x2-\sin^2\frac x2\end{matrix}\right\} \Rightarrow 1-\cos x = 2\sin^2\frac x2$$

g) $\lim_{x\to0}\frac{\operatorname{tg} x}{x} = \lim_{x\to0}\frac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to0} \frac{\sin x}{x}\cdot\frac{1}{\cos x}=1$

h)✓ $\lim_{x\to1}\frac{\sin\pi x}{\sin 3\pi x}$ Rj. $\frac13$ i)✓ $\lim_{n\to\infty}\left(n\sin\frac{\pi}{n}\right)$ Rj. π

j) $\lim_{x\to0}\frac{\sin5x-\sin3x}{\sin x} = \lim_{x\to0}\frac{5\cdot\frac{\sin5x}{5x}-\frac{\sin3x}{3x}\cdot3}{\frac{\sin x}{x}} = 5-3=2$

k) $\lim_{x\to a}\frac{\sin x-\sin a}{x-a} = \lim_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a} = \lim_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\cos\frac{x+a}{2}=\cos a$

$$\left.\begin{matrix} \sin x = \sin\left(\frac{x-a}{2}+\frac{x+a}{2}\right) = \sin\frac{x-a}2\cos\frac{x+a}2+\sin\frac{x+a}2\cos\frac{x-a}2 \\ -\sin a = \sin(-a)=\sin\left(\frac{x-a}2-\frac{x+a}2\right)=\sin\frac{x-a}2\cos\frac{x+a}2-\sin\frac{x+a}2\cos\frac{x-a}2 \\ \sin x-\sin a = 2\sin\frac{x-a}2\cos\frac{x+a}2 \end{matrix}\right\} +$$

6) Izračunati limese

a) $\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x=\lim_{x\to\infty}\left(\frac{\frac{x-1}{x}}{\frac{x+1}{x}}\right)^x=\lim_{x\to\infty}\frac{(1-\frac1x)^x}{(1+\frac1x)^x} = \frac{\lim_{x\to\infty}(1+\frac{-1}{x})^x}{e}=\frac{e^{-1}}{e}=e^{-2}$

b) $\lim_{x\to0}\left(\frac{2+x}{3-x}\right)^x = \left(\frac23\right)^0=1$

c) $\lim_{x\to\infty}\left(\frac{x+1}{2x+1}\right)^{x^2}=\lim_{x\to\infty}\left(\frac{1+\frac1x}{2+\frac1x}\right)^{x^2}=\left(\frac12\right)^{+\infty}=0$

d)✓ $\lim_{x\to1}\left(\frac{x-1}{x^2-1}\right)^{x+1}$ Rj. $\frac14$ e) $\lim_{x\to\infty}\left(\frac1{x^2}\right)^{\frac{2x}{x+1}}$ Rj. 0

Izračunati limes $\lim_{n\to\infty}\left(\frac{1+2+3+\dots+(n-1)}{n+1}-\frac n2\right)$

Rj.

$1+2+3+\dots+(n-1) = \frac{n-1}{2}\left(1+(n-1)\right)$ ← suma aritmetičkog niza

$=\frac{n-1}{2}\cdot n = \frac{n(n-1)}{2}$

$\lim_{n\to\infty}\left(\frac{1+2+3+\dots+(n-1)}{n+1}-\frac n2\right) = \lim_{n\to\infty}\left(\frac{\frac{n(n-1)}{2}}{n+1}-\frac n2\right) =$

$=\lim_{n\to\infty}\left(\frac{n(n-1)}{2(n+1)}-\frac n2\right) = \lim_{n\to\infty}\frac{n(n-1)-n(n+1)}{2(n+1)} =$

$=\lim_{n\to\infty}\frac{n^2-n-n^2-n}{2n+2} = \lim_{n\to\infty}\frac{-2n}{2(n+1)} = \lim_{n\to\infty}\frac{-n \;:n}{n+1\;:n}\left(=\frac{\infty}{\infty}\right)$

$=\lim_{n\to\infty}\frac{-1}{1+\underbrace{\frac1n}_{0}}=-1$

Izračunati $\lim_{x\to1}\frac{\sqrt[3]{x}-1}{1-x}$.

Rj. $a^3-b^3=(a-b)(a^2+ab+b^2)$

$(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)=(\sqrt[3]x)^3-1^3=x-1$

$\lim_{x\to1}\frac{\sqrt[3]x-1}{1-x}\left(\frac00\right) = -\lim_{x\to1}\frac{\sqrt[3]x-1}{x-1}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]x+1)}{(\sqrt[3]{x^2}+\sqrt[3]x+1)} = -\lim_{x\to1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt[3]{x^2}+\sqrt[3]x+1)}$

$=-\lim_{x\to1}\frac1{\sqrt[3]{x^2}+\sqrt[3]x+1}=\frac{-1}{\sqrt[3]1+\sqrt[3]1+1}=-\frac13$

Izračunati limese

a) $\lim\limits_{n\to\infty}\left(1+\frac{a}{n}\right)^{n}$; b) $\lim\limits_{x\to 0}\sqrt[x]{1-2x}$;

c) $\lim\limits_{t\to\infty}\left(\frac{t-3}{t+2}\right)^{2t+1}$; d) $\lim\limits_{x\to\frac{\pi}{4}}(\operatorname{tg}x)^{\operatorname{tg}2x}$

Rj. Znamo da je $\boxed{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e}$ $\boxed{\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e}$

a) $$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n}=\left|\begin{array}{ll}\text{uvodimo smjenu} & \\ \frac{n}{a}=x & n\to\infty \Rightarrow x\to\infty \\ n=ax & \end{array}\right|=$$

$$=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{ax}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{a}=$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{a}=e^{a}$$

b) $$\lim_{x\to 0}\sqrt[x]{1-2x}=\lim_{x\to 0}(1-2x)^{\frac{1}{x}}=\left|\begin{array}{l}\text{uvodimo smjenu} \\ -2x=t \\ x\to 0 \Rightarrow t\to 0\end{array}\right|=$$

$$=\lim_{t\to 0}(1+t)^{-\frac{2}{t}}=\left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{-2}=e^{-2}$$

c) $$\lim_{t\to\infty}\left(\frac{t-3}{t+2}\right)^{2t+1}=\lim_{t\to\infty}\left(\frac{t+2-5}{t+2}\right)^{2t+1}=\lim_{t\to\infty}\left(1+\frac{-5}{t+2}\right)^{2t+1}$$

$$=\left|\begin{array}{ll}\text{uvodimo smjenu} & \\ -\frac{5}{t+2}=x & t\to\infty \Rightarrow x\to 0 \\ -5=x(t+2) & 2t+4=-\frac{10}{x} \\ -\frac{5}{x}=t+2 & 2t+1=-\frac{10}{x}-3\end{array}\right|=\lim_{x\to\infty}(1+x)^{-\frac{10}{x}-3}=$$

$$=\lim_{x\to 0}\left[\left((1+x)^{\frac{1}{x}}\right)^{-10}\cdot(1+x)^{-3}\right]=$$

$$=\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{-10}\cdot\lim_{x\to 0}(1+x)^{-3}=e^{-10}\cdot 1=\frac{1}{e^{10}}$$

d) $$\lim_{x\to\frac{\pi}{4}}(\operatorname{tg}x)^{\operatorname{tg}2x}=\left|\begin{array}{l}\text{uvodimo smjenu} \\ \operatorname{tg}x=1+t \\ x\to\frac{\pi}{4} \Rightarrow t\to 0\end{array}\right.$$

$\operatorname{tg}x=1+t$

$\operatorname{tg}2x=\frac{\sin 2x}{\cos 2x}=$

$=\frac{2\sin x\cos x}{\cos^2x-\sin^2x}\ \begin{array}{l}/:\cos^2x \\ /:\cos^2x\end{array}$

$$\operatorname{tg}2x=\frac{2\operatorname{tg}x}{1-\operatorname{tg}^2x}=\frac{2(1+t)}{1-(1+t)^2}=\frac{2(1+t)}{1-(1+2t+t^2)}=-\frac{2(t+1)}{t(t+2)}\ \Bigg|=$$

$$=\lim_{t\to 0}(1+t)^{-\frac{2(t+1)}{t(t+2)}}=\lim_{t\to 0}\left[(1+t)^{\frac{1}{t}}\right]^{-\frac{2(t+1)}{t+2}}=e^{-1}$$

zato što $\lim\limits_{t\to 0}\frac{-2(t+1)}{t+2}=\frac{-2}{2}=-1$.

Izračunati $\lim_{x\to-\infty}\left(\frac{x+1}{3x+2}\right)^x$

Rj. Znamo da $\lim u^v = \lim u^{\lim v}$.

$$\lim_{x\to-\infty}\left(\frac{x+1}{3x+2}\right)^x = \lim_{x\to-\infty}\left(\frac{1+\frac{1}{x}}{3+\frac{2}{x}}\right)^x = \left(\frac{1}{3}\right)^{\lim_{x\to-\infty} x} = \left(\frac{1}{3}\right)^{-\infty}$$

$$= 3^{\infty} = \infty$$

Izračunati $\lim_{x\to 0}\left(\frac{1}{\sin 2x \sin x} - \frac{1}{2\sin^2 x}\right)$.

Rj.

$$\frac{1}{\sin 2x \sin x} - \frac{1}{2\sin^2 x} = \frac{2\sin x - \sin 2x}{2\sin 2x \sin^2 x} =$$

$$= \frac{2\sin x - 2\sin x\cos x}{2\cdot 2\sin x\cos x\,(1-\cos^2 x)} = \frac{\cancel{2\sin x}\,\cancel{(1-\cos x)}}{2\cdot\cancel{2\sin x}\cos x\,\cancel{(1-\cos x)}(1+\cos x)}$$

$$= \frac{1}{2}\cdot\frac{1}{\cos x(1+\cos x)}$$

$$\lim_{x\to 0}\left(\frac{1}{\sin 2x \sin x} - \frac{1}{2\sin^2 x}\right) = \frac{1}{2}\lim_{x\to 0}\frac{1}{\cos x(1+\cos x)} =$$

$$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

Jednostrani limesi

Ako je $x<a$ i $x\to a$, tada po dogovoru pišemo $x\to a-0$; analogno, ako je $x>a$ i $x\to a$, pišemo to ovako $x\to a+0$.

Brojeve $f(a-0)=\lim\limits_{x\to a-0} f(x)$ i $f(a+0)=\lim\limits_{x\to a+0} f(x)$

nazivamo lijevi limes f-je $f(x)$ u tački a i desni limes f-je $f(x)$ u tački a (ako ti brojevi postoje).

Koriste se i sljedeće dvije oznake

$$f(a+)=\lim_{x\to a+} f(x) \quad \text{i} \quad f(a-)=\lim_{x\to a-} f(x)$$

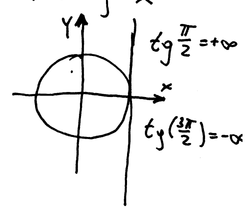
Za postojanje limesa f-je $f(x)$ kada $x\to a$ potrebno je i dovoljno da vrijedi jednakost $f(a-0)=f(a+0)$.

1. Izračunati desni i lijevi limes f-je $f(x)=\operatorname{arc\,tg}\frac{1}{x}$

Rj: $f(+0)=\lim\limits_{x\to +0}\operatorname{arc\,tg}\frac{1}{x}=\frac{\pi}{2}$

$f(-0)=\lim\limits_{x\to -0}\operatorname{arc\,tg}\frac{1}{x}=-\frac{\pi}{2}$

limes f-je $f(x)$ kad $x\to 0$ u ovom slučaju ne postoji



2. Izračunati jednostrane limese

a) $\lim\limits_{x\to -0}\frac{1}{1+e^{\frac{1}{x}}}=\frac{1}{1+e^{-\infty}}=\frac{1}{1+\frac{1}{e^{\infty}}}=1$

b) $\lim \frac{1}{1+e^{\frac{1}{x}}}$ Rj: 0

c) $\lim\limits_{x\to 2+0}\frac{x}{x-2}=\frac{2+0}{2+0-2}=\frac{2+0}{+0}=+\infty$

d) $\lim\limits_{x\to 2-0}\frac{x}{x-2}$ Rj: $-\infty$

e) $\lim\limits_{x\to -0}\frac{|\sin x|}{x}=\lim\limits_{x\to -0}\frac{-\sin x}{x}=-1$

f) $\lim\limits_{x\to +0}\frac{|\sin x|}{x}$ Rj: 1

g) $\lim\limits_{x\to 1-0}\frac{x-1}{|x-1|}=\lim\limits_{x\to 1-0}\frac{(x-1)}{-(x-1)}=\lim\limits_{x\to 1-0}(-1)=-1$

h) $\lim\limits_{x\to 1+0}\frac{x-1}{|x-1|}$ Rj: 1

i) $\lim\limits_{x\to -\infty}\frac{x}{\sqrt{x^2+1}}=\lim\limits_{x\to -\infty}\frac{x}{\sqrt{x^2}}=\lim\limits_{x\to -\infty}\frac{x}{|x|}=\lim\limits_{x\to -\infty}-\frac{x}{x}=-1$

j) $\lim\limits_{x\to +\infty}\frac{x}{\sqrt{x^2+1}}$ Rj: 1

Granična vrijednost funkcije

Neka je data realna funkcija $f: R\to R$;

Pojam granične vrijednosti funkcije

Za neku funkciju $y=f(x)$ kažemo da ima graničnu vrijednost A u tački a ako je

$$\begin{matrix}|f(x)-A|<\varepsilon \\ |x-a|<\delta(\varepsilon)\end{matrix} \quad \text{i pišemo: } \lim_{x\to a} f(x)=A$$

Nreka su $f(x)$ i $g(x)$ i $\lim\limits_{x\to a} f(x)=A$ i $\lim\limits_{x\to a} g(x)=B$ tada važi:

$$\lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)=A\pm B$$

$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)=A\cdot B$$

$$\lim_{x\to a}(c\cdot f(x))=c\cdot\lim_{x\to a} f(x)=cA$$

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\frac{A}{B}$$

$$\lim_{n\to+\infty}(a_n)^k=\left(\lim_{n\to+\infty} a_n\right)^k=a^k \quad k\neq\pm\infty$$

$$\lim_{n\to+\infty}\sqrt[k]{a_n}=\sqrt[k]{\lim_{n\to+\infty} a_n}=\sqrt[k]{a} \quad k\neq\pm\infty$$

$$\lim_{n\to+\infty} k^{a_n}=k^{\lim\limits_{n\to+\infty} a_n}=k^a \quad k\neq\pm\infty$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \qquad \lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{a^x-1}{x}=\ln a$$

ZADACI

1. Naći:

$$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$$

Rješenje

$$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim_{x\to+\infty}\frac{\frac{\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}}=\lim_{x\to+\infty}\frac{1}{\sqrt{\frac{x+\sqrt{x+\sqrt{x}}}{x}}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}}}=$$

$$\lim_{x\to+\infty}\frac{1}{\sqrt{1+\sqrt{\frac{x+\sqrt{x}}{x^2}}}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{\sqrt{x}}{x^2}}}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}=\underline{\underline{1}}$$

2. Naći:

$$\lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3}$$

Rješenje

$$\lim_{x\to a}\frac{x^2-(a+1)x+a}{x^3-a^3}=\lim_{x\to a}\frac{x^2-ax-x+a}{(x-a)(x^2+ax+a^2)}=\lim_{x\to a}\frac{x(x-a)-(x-a)}{(x-a)(x^2+ax+a^2)}=$$

$$\lim_{x\to a}\frac{x-1}{x^2+ax+a^2}=\underline{\underline{\frac{a-1}{3a^2}}}$$

3. Naći:

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

Rješenje

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}\cdot\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}\cdot\frac{\sqrt[4]{x}+1}{\sqrt[4]{x}+1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{x-1}{x-1}\cdot\frac{(\sqrt[4]{x}+1)(\sqrt{x}+1)}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=$$

$$\lim_{x\to 1}\frac{(\sqrt[4]{x}+1)(\sqrt{x}+1)}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}=\frac{4}{\underline{\underline{3}}}$$

4. Naći:

$$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$$

Rješenje

$$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}=$$

$$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{(x-3)(x-1)}\cdot\frac{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}=$$

$$\lim_{x\to 3}\frac{-4x+12}{(x-3)(x-1)\left(\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}\right)}=\lim_{x\to 3}\frac{-4}{(x-1)\left(\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}\right)}=$$

$$\frac{-4}{2(3+3)}=-\frac{1}{\underline{\underline{3}}}$$

5. Naći

$$\lim_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{\pi-3x}$$

Rješenje

$$\lim_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{\pi-3x}=\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{3}}\frac{2\left(\frac{1}{2}-\cos x\right)}{\pi-3x}=\lim_{x\to\frac{\pi}{3}}\frac{2\left(\cos\frac{\pi}{3}-\cos x\right)}{\pi-3x}=$$

$$\lim_{x\to\frac{\pi}{3}}\frac{2\left(-2\sin\frac{\frac{\pi}{3}+x}{2}\sin\frac{\frac{\pi}{3}-x}{2}\right)}{\pi-3x}=\lim_{x\to\frac{\pi}{3}}\frac{-4\sin\frac{\pi+3x}{6}\sin\frac{\pi-3x}{6}}{6\cdot\frac{\pi-3x}{6}}=\frac{-4\sin\frac{\pi+\pi}{6}}{6}=\underline{\underline{-\frac{\sqrt{3}}{3}}}$$

6. Naći

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}$$

Rješenje

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi x}{2}\right)}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}=\lim_{x\to 1}\frac{\sin\left[\frac{\pi}{2}(1-x)\right](1+\sqrt{x})}{1-x}=$$

$$\lim_{x\to 1}\frac{\sin\left[\frac{\pi}{2}(1-x)\right](1+\sqrt{x})}{\frac{2}{\pi}\cdot\frac{\pi}{2}(1-x)}=\frac{1\cdot 2}{\frac{2}{\pi}}=\underline{\underline{\pi}}$$

7. Naći:

$$\lim_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{\sqrt{x}-x\sqrt{x}}{1-\sqrt{x}}}$$

Rješenje

$$\lim_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{\sqrt{x}-x\sqrt{x}}{1-\sqrt{x}}}=\lim_{x\to+\infty}\left(\frac{2+x-1}{2+x}\right)^{\frac{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}}}=\lim_{x\to+\infty}\left(1-\frac{1}{2+x}\right)^{\sqrt{x}(1+\sqrt{x})}=$$

$$\lim_{x\to+\infty}\left\{\left[\left(1-\frac{1}{2+x}\right)^{2+x}\right]^{\frac{1}{2+x}}\right\}^{x+\sqrt{x}}=\left(e^{-1}\right)^{\lim\limits_{x\to+\infty}\frac{x+\sqrt{x}}{2+x}}=\left(e^{-1}\right)^{\lim\limits_{x\to+\infty}\frac{1+\sqrt{\frac{1}{x}}}{1+\frac{2}{x}}}=\underline{\underline{e^{-1}}}$$

8. Naći

$$\lim_{x\to 0}\left(\frac{1+tgx}{1+\sin x}\right)^{\frac{1}{\sin x}}$$

Rješenje

$$\lim_{x\to 0}\left(\frac{1+\mathrm{tgx}}{1+\sin x}\right)^{\frac{1}{\sin x}}=\lim_{x\to 0}\left(\frac{1+\sin x-\sin x+\mathrm{tgx}}{1+\sin x}\right)^{\frac{1}{\sin x}}=\lim_{x\to 0}\left(1-\frac{\sin x-\mathrm{tgx}}{1+\sin x}\right)^{\frac{1}{\sin x}}=$$

$$\lim_{x\to 0}\left[\left(1-\frac{\sin x-\mathrm{tgx}}{1+\sin x}\right)^{\frac{1+\sin x}{\sin x-\mathrm{tgx}}}\right]^{\frac{\sin x-\mathrm{tgx}}{1+\sin x}\cdot\frac{1}{\sin x}}=e^{-\lim\limits_{x\to 0}\frac{\sin x-\mathrm{tgx}}{1+\sin x}\cdot\frac{1}{\sin x}}=e^{-\lim\limits_{x\to 0}\frac{1-\frac{1}{\cos x}}{1+\sin x}}=e^0=\underline{\underline{1}}$$

******moguće su štamparske greške******

225. $$\lim_{x\to\infty}\frac{x^2-3x+1}{2x^2+x}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(x^2-3x+1)/x^2}{(2x^2+x)/x^2}=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{1}{x^2}}{2+\frac{1}{x}}=\frac{1}{2}.$$

226. $$\lim_{x\to\infty}\frac{3x^4-5x^2+7x}{x^4-x^3+5}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(3x^4-5x^2+7x)/x^4}{(x^4-x^3+5)/x^4}=\lim_{x\to\infty}\frac{3-\frac{5}{x^2}+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{5}{x^4}}=3.$$

227. $$\lim_{x\to\infty}\frac{5x^2-x+3}{3x^3+2x-4}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(5x^2-x+3)/x^3}{(3x^3+2x-4)/x^3}=\lim_{x\to\infty}\frac{\frac{5}{x}-\frac{1}{x^2}+\frac{3}{x^3}}{3+\frac{2}{x^2}-\frac{4}{x^3}}=\frac{0}{3}=0.$$

228. $$\lim_{x\to\infty}\frac{6x^4-2x^3+x^2}{2x^3+x^2-3}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(6x^4-2x^3+x^2)/x^4}{(2x^3+x^2-3)/x^4}=\lim_{x\to\infty}\frac{6-\frac{2}{x}+\frac{1}{x^2}}{\frac{2}{x}+\frac{1}{x^2}-\frac{3}{x^4}}=\frac{6}{0}=\infty.$$

229. $$\lim_{x\to\infty}\frac{\sqrt{x^2-2x}}{x+1}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(\sqrt{x^2-2x})/x}{(x+1)/x}=\lim_{x\to\infty}\frac{\sqrt{1-\frac{2}{x}}}{1+\frac{1}{x}}=1.$$

230. $$\lim_{x\to\infty}\frac{x+\sqrt{x^2-x}}{2x+3}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(x+\sqrt{x^2-x})/x}{(2x+3)/x}=\lim_{x\to\infty}\frac{1+\sqrt{1-\frac{1}{x}}}{2+\frac{3}{x}}=\frac{1+1}{2}=1.$$

231. $$\lim_{x\to\infty}\frac{\sqrt{x+3}+\sqrt[4]{x^2-3x+1}}{2\sqrt{x-4}+\sqrt[4]{x^2-5}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{[\sqrt{x+3}+\sqrt[4]{x^2-3x+1}]/\sqrt{x}}{[2\sqrt{x-4}+\sqrt[4]{x^2-5}]/\sqrt{x}}=$$
$$=\lim_{x\to\infty}\frac{\sqrt{1+\frac{3}{x}}+\sqrt[4]{1-\frac{3}{x}+\frac{1}{x^2}}}{2\sqrt{1-\frac{4}{x}}+\sqrt[4]{1-\frac{5}{x^2}}}=\frac{1+1}{2+1}=\frac{2}{3}.$$

232. $$\lim_{x\to\infty}\frac{\sqrt[5]{x^5-2x^3+4}+(3x-4)}{\sqrt[3]{x^3+x^2-4}+\sqrt{x^2-1}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{[\sqrt[5]{x^5-2x^3+4}+(3x-4)]/x}{[\sqrt[3]{x^3+x^2-4}+\sqrt{x^2-1}]/x}=$$
$$=\lim_{x\to\infty}\frac{\sqrt[5]{1-\frac{2}{x^2}+\frac{4}{x^5}}+3-\frac{4}{x}}{\sqrt[3]{1+\frac{1}{x}-\frac{4}{x^3}}+\sqrt{1-\frac{1}{x^2}}}=\frac{1+3}{1+1}=\frac{4}{2}=2.$$

233. $$\lim_{x\to-\infty}\frac{\sqrt{x^2-2x}}{x+1}=\tfrac{\infty}{-\infty}=|\,x\to(-x)\,|=\lim_{x\to\infty}\frac{\sqrt{(-x)^2-2(-x)}}{-x+1}=\lim_{x\to\infty}\frac{\sqrt{x^2+2x}}{-x+1}=$$
$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+2x})/x}{(-x+1)/x}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{2}{x}}}{-1+\frac{1}{x}}=-1.$$

234. $$\lim_{x\to-\infty}\frac{x-\sqrt{x^2+3x}}{2x+1}=\tfrac{-\infty}{-\infty}=|\,x\to(-x)\,|=\lim_{x\to\infty}\frac{(-x)-\sqrt{(-x)^2+3(-x)}}{2(-x)+1}=$$
$$=\lim_{x\to\infty}\frac{-x-\sqrt{x^2-3x}}{-2x+1}=\lim_{x\to\infty}\frac{[-x-\sqrt{x^2-3x}]/x}{[-2x+1]/x}=\lim_{x\to\infty}\frac{-1-\sqrt{1-\frac{3}{x}}}{-2+\frac{1}{x}}=\frac{-1-1}{-2}=1.$$

➢ ZADACI ZA VJEŽBU

235. $$\lim_{x\to\infty}\frac{2x^3-x^2+1}{x^3+2x^2-4}.$$

236. $$\lim_{x\to\infty}\frac{4x^2-x+10}{x^3+x^2-1}.$$

237. $$\lim_{x\to\infty}\frac{2x-1+\sqrt{x^2-x}}{3x+\sqrt{x^2+7}}.$$

238. $$\lim_{x\to-\infty}\frac{2x-1+\sqrt{x^2-x}}{3x+\sqrt{x^2+7}}.$$

239. $$\lim_{x\to\infty}\frac{\sqrt[3]{x^3-2x}+4x-1}{2\sqrt{x^2+3x}+x}.$$

240. $$\lim_{x\to-\infty}\frac{\sqrt[3]{x^3-2x}+4x-1}{2\sqrt{x^2+3x}+x}.$$

241. $$\lim_{x\to\infty}\frac{4\cdot\sqrt{x^2-5x+1}+\sqrt[3]{x^3+x^2-7}}{3x+4+\sqrt{x^2-2x+5}}.$$

242. $\lim\limits_{x\to-\infty}\dfrac{4\cdot\sqrt{x^2-5x+1}+\sqrt[3]{x^3+x^2-7}}{3x+4+\sqrt{x^2-2x+5}}.$

➢ RJEŠENJA

$R235.\ \ 2.\quad R236.\ \ 0.\quad R237.\ \ \dfrac{3}{4}.\quad R238.\ \ \dfrac{1}{2}.\quad R239.\ \ \dfrac{5}{3}.\quad R240.\ \ -5.$

$R241.\ \ \dfrac{5}{4}.\quad R242.\ \ -\dfrac{3}{2}.$

➢ 4.2 NEODREĐENI OBLIK $\infty-\infty$

U ovoj točki ćemo računati limese funkcija kod kojih se nakon uvrštavanja $x=\infty$ pojavljuje neodređeni oblik $\infty-\infty$. U tom slučaju je potrebno danu funkciju transformirati raznim "trikovima" (racionaliziranje, faktoriziranje, itd.) na oblik $\frac{\infty}{\infty}$, te nastaviti u smislu prelaza sa beskonačno velikih na konačne i proizvoljno male veličine (dijeljenje brojnika i nazivnika sa najvećom potencijom), što je objašnjeno u prethodnom poglavlju.

➢ RJEŠENI PRIMJERI

U slijedećim zadacima izračunati limese funkcija.

243. $\lim\limits_{x\to\infty}(\sqrt{x}-\sqrt{x-3})=\lim\limits_{x\to\infty}(\sqrt{x}-\sqrt{x-3})\dfrac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}+\sqrt{x-3}}=\lim\limits_{x\to\infty}\dfrac{x-x+3}{\sqrt{x}+\sqrt{x-3}}=$

$$=\lim_{x\to\infty}\frac{3}{\sqrt{x}+\sqrt{x-3}}=0.$$

244. $\lim\limits_{x\to\infty}(x-\sqrt{x^2-3x+4})=\lim\limits_{x\to\infty}(x-\sqrt{x^2-3x+4})\dfrac{x+\sqrt{x^2-3x+4}}{x+\sqrt{x^2-3x+4}}=$

$$=\lim_{x\to\infty}\frac{x^2-x^2+3x-4}{x+\sqrt{x^2-3x+4}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(3x-4)/x}{[x+\sqrt{x^2-3x+4}]/x}=\lim_{x\to\infty}\frac{3-\dfrac{4}{x}}{1+\sqrt{1-\dfrac{3}{x}+\dfrac{4}{x^2}}}=\frac{3}{2}.$$

245. $\lim\limits_{x\to\infty}(\sqrt{x^2-4}-\sqrt{x^2-2x+5})=\lim\limits_{x\to\infty}(\sqrt{x^2-4}-\sqrt{x^2-2x+5})\dfrac{\sqrt{x^2-4}+\sqrt{x^2-2x+5}}{\sqrt{x^2-4}+\sqrt{x^2-2x+5}}=$

$$=\lim_{x\to\infty}\frac{x^2-4-x^2+2x-5}{\sqrt{x^2-4}+\sqrt{x^2-2x+5}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(2x-9)/x}{[\sqrt{x^2-4}+\sqrt{x^2-2x+5}]/x}=$$

$$=\lim_{x\to\infty}\frac{2-\dfrac{9}{x}}{\sqrt{1-\dfrac{4}{x^2}}+\sqrt{1-\dfrac{2}{x}+\dfrac{5}{x^2}}}=\frac{2}{1+1}=1.$$

246. $\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x}(\sqrt{x-3}-\sqrt{x+4})}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x}(\sqrt{x-3}-\sqrt{x+4})}\dfrac{\sqrt{x-3}+\sqrt{x+4}}{\sqrt{x-3}+\sqrt{x+4}}=$

$$=\lim_{x\to\infty}\frac{\sqrt{x-3}+\sqrt{x+4}}{\sqrt{x}(x-3-x-4)}=-\frac{1}{7}\lim_{x\to\infty}\frac{\sqrt{x-3}+\sqrt{x+4}}{\sqrt{x}}=\tfrac{\infty}{\infty}=$$

$$=-\frac{1}{7}\lim_{x\to\infty}\frac{[\sqrt{x-3}+\sqrt{x+4}]/\sqrt{x}}{\sqrt{x}/\sqrt{x}}=-\frac{1}{7}\lim_{x\to\infty}[\sqrt{1-\frac{3}{x}}+\sqrt{1+\frac{4}{x}}]=-\frac{2}{7}.$$

247. $\lim\limits_{x\to-\infty}(x+\sqrt{x^2-x+2})=\ |x\to(-x)|\ =\lim\limits_{-x\to-\infty}[(-x)+\sqrt{(-x)^2-(-x)+2}\,]=$

$$=\lim_{x\to\infty}[-x+\sqrt{x^2+x+2}\,]=\lim_{x\to\infty}[-x+\sqrt{x^2+x+2}\,]\frac{x+\sqrt{x^2+x+2}}{x+\sqrt{x^2+x+2}}=$$

$$=\lim_{x\to\infty}\frac{-x^2+x^2+x+2}{x+\sqrt{x^2+x+2}}=\tfrac{\infty}{\infty}=\lim_{x\to\infty}\frac{(x+2)/x}{[x+\sqrt{x^2+x+2}]/x}=$$

$$=\lim_{x\to\infty}\frac{1+\dfrac{2}{x}}{1+\sqrt{1+\dfrac{1}{x}+\dfrac{2}{x^2}}}=\frac{1}{1+1}=\frac{1}{2}.$$

➢ ZADACI ZA VJEŽBU

248. $\lim\limits_{x\to-\infty}(x+\sqrt{x^2-3x+1}).$

249. $\lim\limits_{x\to\infty}(\sqrt{x^2-3x+1}-x).$

250. $\lim\limits_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2+5x}).$

251. $\lim\limits_{x\to\infty}(x-\sqrt[3]{x^3+2x^2-1})$.

252. $\lim\limits_{x\to\infty}(\sqrt[4]{x^4+x^3-2}-\sqrt[4]{x^4-x^2+3x})$.

➢ RJEŠENJA

$R248.\ \frac{3}{2}.\quad R249.\ -\frac{3}{2}.\quad R250.\ -2.\quad R251.\ -\frac{2}{3}.\quad R252.\ \frac{1}{4}.$

➢ 4.3 NEODREĐENI OBLIK 1^∞

U ovoj točki računamo limese funkcija oblika $y=f(x)^{g(x)}$ kod kojih nakon uvrštavanja $x=\infty$ dobivamo oblik 1^∞. Osim svojstava limesa, nabrojanih na početku ovog poglavlja, koristit ćemo važan identitet:

$$\boxed{\lim_{x\to\infty}(1+\frac{1}{x})^x=e.}$$

Nadalje, treba primijeniti određene «trikove» pomoću kojih se dani oblik $y=f(x)^{g(x)}$ transformira na eksponencijalni oblik $e^{\frac{\infty}{\infty}}$, pa potom u eksponentu primijeniti rješavanje oblika $\frac{\infty}{\infty}$ s početka ovog poglavlja.

➢ RJEŠENI PRIMJERI

U slijedećim zadacima izračunati limese funkcija.

253. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x}\right)^x=1^\infty=\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}3}=e^3.$

254. $\lim\limits_{x\to\infty}\left(\frac{x+a}{x}\right)^x=1^\infty=\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{a}}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{a}}\right)^{\frac{x}{a}a}=$

$$=\left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{a}}\right)^{\frac{x}{a}}\right)^a=e^a.$$

255. $\lim\limits_{x\to\infty}\left(\frac{x}{x+4}\right)^x=1^\infty=\lim\limits_{x\to\infty}\frac{1}{\left(\frac{x+4}{x}\right)^x}=\frac{1}{\lim\limits_{x\to\infty}\left(\frac{x+4}{x}\right)^x}=\frac{1}{e^4}=e^{-4}.$

256. $\lim\limits_{x\to\infty}\left(\frac{x}{x+a}\right)^x=1^\infty=\lim\limits_{x\to\infty}\frac{1}{\left(\frac{x+a}{x}\right)^x}=\frac{1}{\lim\limits_{x\to\infty}\left(\frac{x+a}{x}\right)^x}=\frac{1}{e^a}=e^{-a}.$

257. $\lim\limits_{x\to\infty}\left(\frac{x^2+5}{x^2}\right)^{x^2}=1^\infty=\lim\limits_{x\to\infty}\left(1+\frac{5}{x^2}\right)^{x^2}=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x^2}{5}}\right)^{x^2}=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x^2}{5}}\right)^{\frac{x^2}{5}5}=e^5.$

258. $\lim\limits_{x\to\infty}\left(\frac{x-3}{x+1}\right)^x=1^\infty=\lim\limits_{x\to\infty}\left(1+\frac{x-3}{x+1}-1\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{-4}{x+1}\right)^x=$

$$=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x+1}{-4}}\right)^{\frac{x+1}{-4}\left(\frac{-4x}{x+1}\right)}=e^{\lim\limits_{x\to\infty}\frac{-4x}{x+1}}=e^{-4}.$$

259. $\lim\limits_{x\to\infty}\left(\frac{x-a}{x-b}\right)^x=1^\infty=\lim\limits_{x\to\infty}\left(1+\frac{x-a}{x-b}-1\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{b-a}{x-b}\right)^x=$

$$=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-b}{b-a}}\right)^{\frac{x-b}{b-a}\left(\frac{(b-a)x}{x-b}\right)}=e^{(b-a)\lim\limits_{x\to\infty}\frac{x}{x-b}}=e^{b-a}.$$

➢ ZADACI ZA VJEŽBU

260. $\lim\limits_{x\to\infty}\left(\frac{\sqrt{x}+2}{\sqrt{x}-1}\right)^{2\sqrt{x}}$.

261. $\lim\limits_{x\to\infty}\left(\frac{x^3+x}{x^3+4}\right)^{3x^2}$.

262. $\lim\limits_{x\to\infty}\left(\frac{x^2-x}{x^2+3x-1}\right)^x$.

263. $\lim\limits_{x\to\infty}\left(\frac{x-2\sqrt{x}+3}{x+\sqrt{x}-1}\right)^{\sqrt{x}}$.

LIMES NIZA

Def: $\lim\limits_{n\to\infty} a_n = a$ ako $(\forall \varepsilon>0)(\exists n_0=n_0(\varepsilon)\in\mathbb{N})$ tako da

je $n\geq n_0 \Rightarrow |a_n - a| < \varepsilon$

razlika vrlo mala blizu nule

ε - ima jako puno def. i dokaza teorema

najčešće ga koriste kada ga hoće označiti kao pozitivan broj, ali malo veći od 0. Znači ne nula, nego malo veći.

$a_n \approx a$

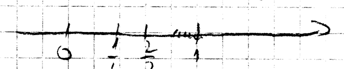
za velike indekse n - što je veći indeks bliži smo limesu.

Pr. $a_n = \frac{1}{n+1}$ $\quad \lim\limits_{n\to\infty} a_n = 1$

$a_3 = \frac{3}{4}$; $\quad \left|\frac{3}{4} - 1\right| = \frac{1}{4}$

$a_{10} = \frac{10}{11}$; $\quad \left|\frac{10}{11} - 1\right| = \frac{1}{11}$ (manji od $\frac{1}{4}$)

Što je veći indeks n član niza je sve bliži tom limesu.



što smo bliže jedinici razlika između člana i nule je manja.

Limesu se može približavati na 2 načina

kada raste sa lijeva na desno

kada opada sa desna na lijevo

konvergentan niz ne mora biti konvergentan, ali je svaki put bliže limesu a

- kod limesa polinoma kada n teži ∞, uvijek će rezultat biti ∞ samo pitanje $+$ ili $-$.

- Najveći stepen ima i najbrži rast u ∞

1. $\lim\limits_{n\to\infty} (n^2 + 9n - 7) = \lim\limits_{n\to\infty} n^2 = +\infty$

2. $\lim\limits_{n\to\infty} (2n^2 - n^3 - 5n + 11) = \lim\limits_{n\to\infty} (-n^3) = -\infty$

3. $\lim\limits_{n\to\infty} \frac{4n^3 + 2n^2 - 1}{3n^3 + 7} = \lim\limits_{n\to\infty} \frac{4n^3}{3n^3} = \frac{4}{3}$

dijelimo s polinomom istog stepena, odnosno s najvećim stepenom

4. $\lim\limits_{n\to\infty} \frac{2n+1}{n^2-5} = \lim\limits_{n\to\infty} \frac{2n}{n^2} = \frac{2}{n} = 0$, kada u brojniku imamo konstantu, a u nazivniku

$\frac{c}{\infty} = 0$ (konstanta) (c = constanta)

Ukoliko dobijemo nule u brojniku ili nazivniku, tada jednačina nema pravo

5. $\lim\limits_{n\to\infty} \frac{1+2+3+\dots+n}{n^2}$ suma aritmetičkog niza

$= \lim\limits_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2}$

$= \lim\limits_{n\to\infty} \frac{n(n+1)}{2n^2}$

$= \lim\limits_{n\to\infty} \frac{\cancel{n}+1}{2n}$

$= \lim\limits_{n\to\infty} \frac{n}{2n} = \frac{1}{2}$

sa ispita

6. $\lim\limits_{n\to\infty} \frac{1^2+2^2+3^2+\dots+(3n+2)^2}{8n^3-(2n-1)^3} = S_2(n)$

$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ NAPAMET!!!

$1^2 + 2^2 + \dots + (3n+2)^2 = \frac{(3n+2)(3n+2+1)\cdot[2\cdot(3n+2)+1]}{6}$

$\lim\limits_{n\to\infty} \frac{\frac{(3n+2)(3n+3)(6n+5)}{6}}{8n^3-(8n^3-3\cdot 4n^2+3\cdot 2n-1)}$ u svakoj zagradi uzeti najveći

$= \lim\limits_{n\to\infty} \frac{\frac{3n\cdot 3n\cdot \cancel{6}n}{\cancel{6}}}{\cancel{8n^3}-\cancel{8n^3}+12n^2-6n+1} = \lim\limits_{n\to\infty} \frac{9n^3}{12n^2} = \lim\limits_{n\to\infty} \frac{3n}{4} = +\infty$

7) $\lim\limits_{n\to\infty} \frac{1\cdot2+2\cdot3+3\cdot4+\dots+n(n+1)}{n^3}$

ako je sabira dva ili više, suma od ..., onda rastavljamo na 2 sume

$S(n) = 1\cdot2+2\cdot3+\dots+n(n+1)$

$= \sum\limits_{k=1}^{m} k(k+1)$ = skraćeno napisano $= \sum\limits_{k=1}^{m}(k^2+k) =$

$= \sum\limits_{k=1}^{m} k^2 + \sum\limits_{k=1}^{m} k =$

$= S_2(m)+S_1(m) = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)+3n(n+1)}{6}$

$= \frac{n(n+1)(2n+1+3)}{6} = \frac{n(n+1)(2n+4)}{6}$

$= \frac{2n(n+1)(n+2)}{6} = \frac{n(n+1)(n+2)}{3}$

$\lim\limits_{n\to\infty} \frac{\frac{n(n+1)(n+2)}{3}}{n^3} = \lim\limits_{n\to\infty} \frac{n\cdot n\cdot n}{3n^3} = \frac{1}{3}$ ako ove saberemo izvana

do ovdje - misli odavde

Za vježbu:

a) $\lim\limits_{n\to\infty}\left[\frac{1+3+5+\dots+(2n-1)}{n+1} - \frac{2n+1}{2}\right]$

b) $\lim\limits_{n\to\infty} \frac{1^2+2^2+\dots+(n-1)^2}{n^2}$

c) $\lim\limits_{n\to\infty} \frac{1\cdot2^2+2\cdot3^2+\dots+n(n+1)^2}{1^2\cdot2+2^2\cdot3+\dots+n^2(n+1)}$

Uputa: $\lim\limits_{n\to\infty} \frac{\sum\limits_{k=1}^{n} k(k+1)^2}{\sum\limits_{k=1}^{n} k^2(k+1)}$

d) za ispit

$\lim\limits_{m\to\infty} \frac{5+9+13+\dots+(8n-3)}{2+5+8+\dots+(6n+11)}$ aritmetički nizove, primijeniti formulu za sumu aritmetičkog niza

Postupak racionalizacije → pomnoži konjugat onda razlike kvadrata ... ako ako imamo korjene

8. $\lim\limits_{n\to\infty} \sqrt{n}\left(\overset{+\infty}{\sqrt{n+3}} - \overset{+\infty}{\sqrt{n+1}}\right)$ - ne može se primjeniti postupak izvlačenja konstante zbog minusa (tip: $+\infty-\infty$), može ako je između +, a ako je jer kad se √ sabiraju sabirci to je $+\infty$

$= \lim\limits_{n\to\infty} \sqrt{n}\left(\sqrt{n+3}-\sqrt{n+1}\right)\cdot\frac{\sqrt{n+3}+\sqrt{n+1}}{\sqrt{n+3}+\sqrt{n+1}}$

$= \lim\limits_{n\to\infty} \sqrt{n}\cdot\frac{(\sqrt{n+3})^2-(\sqrt{n+1})^2}{\sqrt{n+3}+\sqrt{n+1}}$

$= \lim\limits_{n\to\infty} \frac{\sqrt{n}(n+3-n-1)}{\sqrt{n+3}+\sqrt{n+1}}$

$= \lim\limits_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \lim\limits_{n\to\infty} \frac{2\sqrt{n}}{2\sqrt{n}} = 1$

odavde se mogu izvući

9. $\lim\limits_{n\to\infty} \left(\underset{a}{n} - \underbrace{\sqrt[3]{n^3-5n^2+4n+7}}_{b}\right) \quad (-\infty, +\infty)$

$= a^3-b^3 = (a-b)\cdot(a^2+ab+b^2)$ vraćamo po ovoj formuli

$= \lim\limits_{n\to\infty} \frac{\left(n-\sqrt[3]{n^3-5n^2+4n+7}\right)\cdot\left[n^2+n\cdot\sqrt[3]{n^3}+\left(\sqrt[3]{n^3}\right)^2\right]}{n^2+n\cdot\sqrt[3]{}+\left(\sqrt[3]{}\right)^2}$

$\lim\limits_{n\to\infty} \frac{n^3-\left(\sqrt[3]{n^3-5n^2+4n+7}\right)^3}{n^2+n\sqrt[3]{}+\left(\sqrt[3]{}\right)^2} = \lim\limits_{n\to\infty} \frac{5n^2-4n-7}{n^2+n\sqrt[3]{}+\left(\sqrt[3]{}\right)^2}$

odaberemo najveći član

$= \lim\limits_{n\to\infty} \frac{5n^2}{n^2+n\cdot\sqrt[3]{n^3}+\left(\sqrt[3]{n^3}\right)^2} = \lim\limits_{n\to\infty} \frac{5n^2}{n^2+n^2+n^2} = \lim\limits_{n\to\infty} \frac{5n^2}{3n^2} = \frac{5}{3}$

Za vježbu:

a) $\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}\right)$

proširit razliku kvadrata

b) $\lim_{n\to\infty}\left(\sqrt[4]{n+1}-\sqrt[4]{n}\right)$ - 2 puta r. k.

c) $\lim_{n\to\infty} n\cdot\left(1-\sqrt[3]{1-\frac{1}{n}}\right)$ sa lupita

d) $\lim_{n\to\infty}\left(n-\sqrt[3]{n^3-n^2+1}\right)$ razlika kubova $\quad \frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$

10. $\lim_{n\to\infty}\Big(\underbrace{\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{n(n+1)}}_{S(n)}\Big)$ svaki razlomak, kao razlika 2 razl.

$$S(n)=\frac{2-1}{1\cdot2}+\frac{3-2}{2\cdot3}+\frac{4-3}{3\cdot4}+\dots+\frac{(n+1)-n}{n(n+1)}$$

$$=\frac{2}{1\cdot2}-\frac{1}{1\cdot2}+\frac{3}{2\cdot3}-\frac{2}{2\cdot3}+\frac{4}{3\cdot4}-\frac{3}{3\cdot4}+\dots+\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}$$

$$=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty}S(n)=1$$

11. $\lim_{n\to\infty}\Big(\underbrace{\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\dots+\frac{1}{(3n-2)(3n+1)}}_{S(n)}\Big)$ ostatak pomoću prethodnog zadatka, ali 3, a u nazivniku.

Uputa: $S(n)=\frac{1}{3}\left[\frac{4-1}{1\cdot4}+\frac{7-4}{4\cdot7}+\dots+\frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)}\right]$

Za vježbu:

a) $\lim_{n\to\infty}\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\dots+\frac{1}{1+2+\dots+n}\right)$

b) $\lim_{n\to\infty}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)\cdots\left(1-\frac{1}{\frac{n(n-1)}{2}}\right)$ [illegible]

c) $\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}}\cdot\left(\frac{1}{1+\sqrt[3]{2}+\sqrt[3]{4}}+\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}+\dots+\frac{1}{\sqrt[3]{(n-1)^2}+\sqrt[3]{n(n-1)}+\sqrt[3]{n^2}}\right)$ racionalizacija

$\lim_{n\to\infty}=0$

d) $\lim_{n\to\infty}\left(\frac{1}{\ln 2\cdot\ln 4}+\frac{1}{\ln 4\cdot\ln 8}+\dots+\frac{1}{\ln 2^{n-1}\cdot\ln 2^n}\right)$ ($\ln 2^2$, $\ln 2^3$)

$$\lim_{n\to\infty}a^n=\begin{cases}0, & -1<a<1\\ 1, & a=1\\ +\infty, & a>1\end{cases}$$

$\lim_{n\to\infty}a^n$ ne postoji ako je $a\le-1$

12. $$\lim_{n\to\infty}\frac{(-2)^n+3^n\ /:3^{n+1}}{(-2)^{n+1}+3^{n+1}\ /:3^{n+1}}=\lim_{n\to\infty}\frac{\frac{(-2)^n}{3^{n+1}}+\frac{3^n}{3^{n+1}}}{\frac{(-2)^{n+1}}{3^{n+1}}+1}$$

$$=\lim_{n\to\infty}\frac{\frac{(-2)^n}{3^n\cdot3}+\frac{1}{3}}{\left(-\frac{2}{3}\right)^{n+1}+1}=\lim_{n\to\infty}\frac{\left(-\frac{2}{3}\right)^n\cdot\frac{1}{3}+\frac{1}{3}}{\left(-\frac{2}{3}\right)^{n+1}+1}=\frac{\frac{1}{3}}{1}=\frac{1}{3}$$

Za vježbu:

$\lim_{n\to\infty}\frac{1+a+a^2+\dots+a^n}{1+b+b^2+\dots+b^n}$, ako je $a^2<b^2<1$

Teorema o lopovom i 2 policajca

Ako je niz $b_n \le a_n \le c_n$ i $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = a$

$\Rightarrow \lim_{n\to\infty} a_n = a$

Zadaci:

1. $\lim_{n\to\infty} \sqrt[n]{2^n+2} = 2$

niz koji teži ka 2

$$\sqrt[n]{2^n} < \sqrt[n]{2^n+2} < \sqrt[n]{\left(2+\frac{1}{n}\right)^n}$$

$\sqrt[n]{2^n} = 2$ — očigledno

$\Leftrightarrow 2^n+2 < \left(2+\frac{1}{n}\right)^n$ nije prirodan broj, pošto su potencije nego je teoremski, pa se skraćivanjem pravila dalje

$\Leftrightarrow \cancel{2^n}+2 < \cancel{2^n} + \cancel{n}\cdot 2^{n-1}\cdot\frac{1}{\cancel{n}} + \cdots$

(2.) uopšte jedan od lagenih

$\lim_{n\to\infty} \frac{\cos(n^2+n)}{n+1}$

$(-1,1)$ nema neke uloge — Ako pod dakle je pod cos ili sin ovo determinirano

$$\frac{-1}{n+1} < \frac{\cos(n^2+n)}{n+1} < \frac{1}{n+1}$$

$\to 0$

svaka suma se ne može svesti; ~~ovl~~

(3) $\lim_{n\to\infty} a_n$, $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$

IDEJA: uoči koji je najveći, a koji najmanji i sa njima neki razlomak zamijeniti

$$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

najveći — zbog recipročne vrijednosti; najmanji

$$\frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} < a_n < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}$$

$$\frac{n}{\sqrt{n^2+n}} < a_n < \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = \underline{1}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2}} = 1$$

$\lim_{n\to\infty} a_n = \underline{1}$

(4.) $\lim_{n\to\infty} \frac{n}{2^n}$

binomna formula

$$2^n = (1+1)^n = 1^n + \binom{n}{1}1^{n-1}\cdot 1 + \binom{n}{2}1^{n-2}\cdot 1^2 + \cdots + 1^n > \binom{n}{2} = \frac{n\cdot(n-1)}{2}$$

zbog recipročne vrijednosti i razmišljanja

$$\Rightarrow \frac{n}{2^n} < \frac{n}{\frac{n(n-1)}{2}} = \frac{2n}{n(n-1)} = \frac{2}{n-1}$$

$$0 < \frac{n}{2^n} < \frac{2}{n-1}$$

$\frac{2}{n-1}$ teži nuli $\to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{n}{2^n} = 0$$

Za vježbu: Dokazati da je $\lim_{n\to\infty} \frac{n}{a^n} = 0$ ako je $a > 1$, pomoću binomne formule

po binomnoj formuli

$a^n = (\underbrace{a-1}_{>0}+1)^n = \ldots$

$> \binom{n}{2}\cdots$

Izvod f-je

Definicija. Neka je f-ja f definisana na otvorenom intervalu (a,b) i neka je $c\in(a,b)$. Kažemo da f ima izvod (ili derivaciju) u tački c ako postoji limes $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. Vrijednost limesa obilježavamo sa $f'(c)$ i zovemo izvod f-je f u tački c.

1. Korištenjem navedene definicije naći izvode u tački c sljedećih f-ja:

a) $y=x$ c) $y=\cos x$ e)ˇ $y=x^2$

b) $y=\sqrt[3]{x}$ d)ˇ $y=x^\alpha,\ \alpha\in\mathbb{R}$ f)ˇ $y=\sin x$

R: a) $f(x)=x$, $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{x-c}{x-c}=\lim_{x\to c}1=1$

$$\Rightarrow (x)'=1$$

b) $f(x)=\sqrt[3]{x}$, $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{\sqrt[3]{x}-\sqrt[3]{c}}{x-c}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}{(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}$

$$=\lim_{x\to c}\frac{\cancel{x-c}}{\cancel{(x-c)}(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}=\frac{1}{3\sqrt[3]{c^2}}\Rightarrow(\sqrt[3]{x})'=\frac{1}{3\sqrt[3]{x^2}}$$

c) $f(x)=\cos x$, $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{\cos x-\cos c}{x-c}\overset{(*)}{=}$

$$\cos x=\cos\frac{x+c+x-c}{2}=\cos\left(\frac{x+c}{2}+\frac{x-c}{2}\right)=\cos\frac{x+c}{2}\cos\frac{x-c}{2}-\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

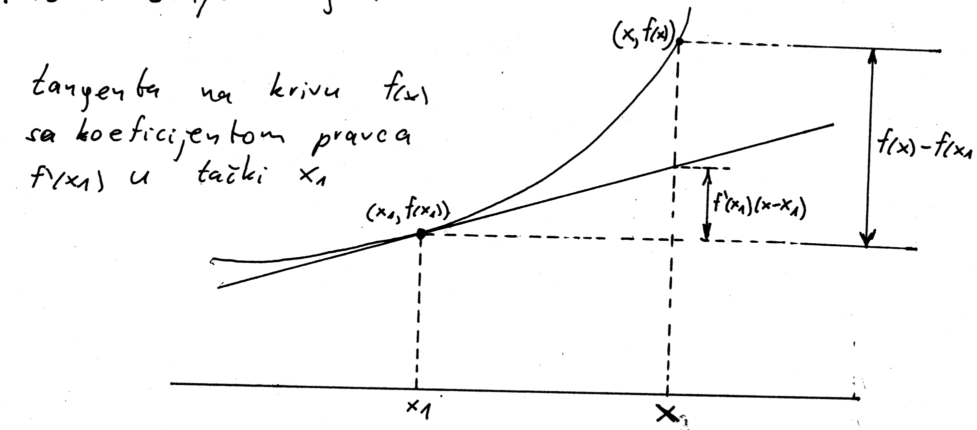
$$\cos c=\cos\frac{x+c-x+c}{2}=\cos\left(\frac{x+c}{2}-\frac{x-c}{2}\right)=\cos\frac{x+c}{2}\cos\frac{x-c}{2}+\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

$$\cos x-\cos c=-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

$$\overset{(*)}{=}\lim_{x\to c}\frac{-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}}{x-c}=-\lim_{x\to c}\sin\frac{x+c}{2}\cdot\underbrace{\lim_{x\to c}\frac{\sin\frac{x-c}{2}}{\frac{x-c}{2}}}_{1}=-\sin c\Rightarrow(\cos x)'=-\sin x$$

Ako f-ja $f(x)$ ima izvod u tački c tada je $f(x)$ neprekidna u tački c.

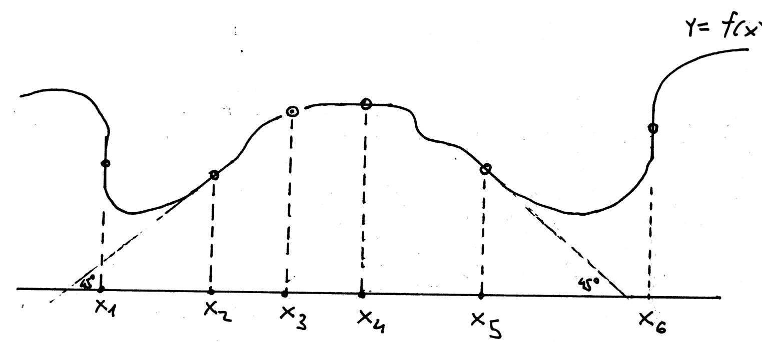
Izvodi se upotrebljavaju u mnogim problemima, a najvažnije dvije skupine su:
1. određivanje brzine tačke koja se kreće pravolinijski
2. iznalaženje tangente na krivu



$$y-y_1=k(x-x_1)$$

$f(x)-f(x_1)=f'(x_1)(x-x_1)$ jednačina tangente na krivu $y=f(x)$ u nekoj tački $(x_1,f(x_1))$

$k_1k_2=-1$ uslov normalnosti dvije prave



$f'(x_1)=-\infty$ $f'(x_3)$ ne postoji $f'(x_5)=-1$

$f'(x_2)=1$ $f'(x_4)=0$ $f'(x_6)=\infty$

Tablica izvoda

1. $c' = 0$, c - konst.
2. $(x^\alpha)' = \alpha x^{\alpha-1}$, $\alpha \in \mathbb{R}$

   $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, $x > 0$
3. $(a^x)' = a^x \ln a$

   $(e^x)' = e^x$
4. $(\log_a x)' = \frac{1}{x \ln a}$

   $(\ln x)' = \frac{1}{x}$
5. $(\sin x)' = \cos x$
6. $(\cos x)' = -\sin x$
7. $(tg\, x)' = \frac{1}{\cos^2 x}$
8. $(ctg\, x)' = -\frac{1}{\sin^2 x}$
9. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $|x|<1$
10. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $|x|<1$
11. $(arc\,tg\, x)' = \frac{1}{1+x^2}$
12. $(arc\,ctg\, x)' = -\frac{1}{1+x^2}$

$sh\, x = \frac{e^x - e^{-x}}{2}$ $\quad ch\, x = \frac{e^x + e^{-x}}{2}$

13. $(sh\, x)' = ch\, x$
14. $(ch\, x)' = sh\, x$
15. $(th\, x)' = \frac{1}{ch^2 x}$
16. $(cth\, x)' = -\frac{1}{sh^2 x}$

Pravila izvoda:

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$
2. $(f \cdot g)'(c) = f'(c) g(c) + f(c) g'(c)$
3. $(\alpha f)'(c) = \alpha f'(c)$
4. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c) g(c) - f(c) g'(c)}{g(c)^2}$, $\quad g(c) \neq 0$

1. Izračunati izvode f-ja:

a) $y = x^5 - 4x^3 + 2x - 3$

Rj. $y' = 5x^4 - 12x^2 + 2$

b) $y = ax^2 + bx + c$

Rj. $y = 2ax + b$

c) $y = -\frac{5x^3}{a}$

Rj. $y' = -\frac{5}{a}(x^3)' = -\frac{15}{a} x^2$

d) $y = x^2 \sqrt[3]{x^2}$

Rj. $y = x^2 \cdot x^{\frac{2}{3}} = x^{\frac{8}{3}}$

$y' = \frac{8}{3} x^{\frac{5}{3}} = \frac{8}{3}\sqrt[3]{x^5} = \frac{8}{3} x \sqrt[3]{x^2}$

e) $y = \frac{a+bx}{c+dx}$

Rj. $y' = \frac{b(c+dx) - (a+bx)\cdot d}{(c+dx)^2}$

$y' = \frac{bc + bdx - ad - bdx}{(c+dx)^2}$

$y' = \frac{bc - ad}{(c+dx)^2}$

f) $y = \frac{2}{2x-1} - \frac{1}{x}$, $\quad$ znamo: $\frac{1}{x} = x^{-1}$

Rj. $y' = \frac{0(2x-1) - 2(2)}{(2x-1)^2} - (-1)x^{-2}$

$y' = \frac{-4}{(2x-1)^2} + \frac{1}{x^2}$

$y' = \frac{1-4x}{x^2(2x-1)^2}$

g) $y = \frac{ax^6 + b}{\sqrt{a^2+b^2}}$

Rj. $y = \frac{a}{\sqrt{a^2+b^2}} x^6 + \frac{b}{\sqrt{a^2+b^2}}$

$y' = \frac{6a}{\sqrt{a^2+b^2}} x^5$

h) $y = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$

Rj. $y' = 3 \cdot \frac{2}{3} x^{-\frac{1}{3}} - 2 \cdot \frac{5}{2} x^{\frac{3}{2}} - 3x^{-4}$

$= 2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}$

i) $y = \frac{2x+3}{x^2-5x+5}$

Rj. $y' = \frac{2(x^2-5x+5) - \overbrace{(2x+3)(2x-5)}^{4x^2-10x+6x-15}}{(x^2-5x+5)^2}$

$y' = \frac{2x^2 - 10x + 10 - 4x^2 + 4x + 15}{(x^2-5x+5)^2}$

$y' = \frac{-2x^2 - 6x + 25}{(x^2-5x+5)^2}$

2. Izračunati izvode f-ja:

a) $y = at^{m} + bt^{m+n}$ Rj. $y' = mat^{m-1} + b(m+n)t^{m+n-1}$

b) $y = \frac{a}{\sqrt[3]{x^2}} - \frac{b}{x\sqrt[3]{x}}$, Rj. $y' = \frac{4b}{3x^2\sqrt[3]{x}} - \frac{2a}{3x\sqrt[3]{x^2}}$

c) $y = \frac{1+\sqrt{z}}{1-\sqrt{z}}$, $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Rj. $$y' = \frac{\frac{1}{2\sqrt{z}}(1-\sqrt{z}) - (1+\sqrt{z})\left(-\frac{1}{2\sqrt{z}}\right)}{(1-\sqrt{z})^2} = \frac{\frac{1-\cancel{\sqrt{z}}+1+\cancel{\sqrt{z}}}{2\sqrt{z}}}{(1-\sqrt{z})^2} = \frac{1}{(1-\sqrt{z})^2\sqrt{z}}$$

d) $y = tg\,x - ctg\,x$

Rj. $$y' = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x} = \frac{4}{(2\sin x\cos x)^2}$$

$$y' = \frac{4}{\sin^2 2x}$$

e) $y = \frac{\pi}{x} + \ln 2$, Rj. $y' = -\frac{\pi}{x^2}$

f) $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

Rj. $$y' = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \frac{-(\sin^2 x - \cancel{2\sin x\cos x} + \cos^2 x + \sin^2 x + \cancel{2\sin x\cos x} + \cos^2 x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-2}{(\sin x - \cos x)^2}$$

g) $y = 2t\sin t - (t^2-2)\cos t$

Rj. $$y' = 2(\sin t + t\cos t) - [2t\cos t + (t^2-2)(-\sin t)] =$$

$$= 2\sin t + \cancel{2t\cos t} - \cancel{2t\cos t} + (t^2-2)\sin t = 2\sin t + (t^2-2)\sin t =$$

$$= 2\sin t + t^2\sin t - 2\sin t$$

$$y' = t^2 \sin t$$

$y = x\arcsin x$

Rj. $y' = \arcsin x + x\cdot\frac{1}{\sqrt{1-x^2}}$

$y = (x-1)e^x$

$\log_B A = \frac{\ln A}{\ln B}$

Rj. $y' = e^x + (x-1)e^x$

$y' = e^x(1+x-1) = xe^x$

$y = \frac{x^5}{e^x}$

Rj. $y' = \frac{5x^4e^x - x^5e^x}{e^{2x}} = \frac{x^4e^x(5-x)}{(e^x)^2}$

$y' = \frac{x^4(5-x)}{e^x}$

$y = \frac{x^2}{\ln x}$

Rj. $y' = \frac{2x\cdot\ln x - x^2\cdot\frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x}$

$y' = \frac{x(2\ln x - 1)}{\ln^2 x}$

$y = \ln x\,\underbrace{\lg x}_{=\log_{10} x} - \ln a\cdot\log_a x$

Rj. $y' = \frac{1}{x}\log_{10} x + \frac{\ln x}{x\ln 10} - \cancel{\ln a}\,\frac{1}{x\cancel{\ln a}}$

$y' = \frac{1}{x}\cdot\frac{\ln x}{\ln 10} + \frac{\ln x}{x\ln 10} - \frac{1}{x}$

$y' = \frac{2\ln x}{x\ln 10} - \frac{1}{x}$

$y = x\,ctg\,x$ Rj. $y' = ctg\,x - \frac{x}{\sin^2 x}$

$y = \frac{(1+x^2)\,arctg\,x - x}{2}$ Rj. $y' = x\,arctg\,x$

$y = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$ Rj. $y' = \frac{2}{x} + \frac{\ln x}{x^2} - \frac{2}{x^2}$

$\log_B A = \frac{\log_a A}{\log_a B}$, $\ln x = \log_e x$, $\log_a B = \frac{1}{\log_B a}$

Izvodi složenih f-ja

$$y=f(g(x)),\quad y'_x=f'_s\cdot g'_x \qquad \text{ili} \qquad \left.\begin{aligned} y&=\psi(u)\\ u&=\varphi(x)\end{aligned}\right\}\ y=\psi(\varphi(x))$$

$$y'_x=y'_u\cdot u'_x$$

1. Naći izvode sljedećih f-ja:

a) $y=(1+3x-5x^2)^{30}$

Rj. $y=u^{30}$, gdje je $u=1+3x-5x^2$

$$y'=30u^{29}\cdot u',\quad u'=3-10x$$

$$y'=30(1+3x-5x^2)^{30}\cdot(3-10x)$$

b) $y=((3+2x^2))^4$

Rj.

$$y'=4(3+2x^2)^3\cdot(3+2x^2)'$$

$$y'=4(3+2x^2)^3\cdot 4x=16x(3+2x^2)^3$$

c) $y=\sqrt[3]{a+bx^3}$

Rj. $y=\sqrt[3]{u}$, $u=a+bx^3$

$$y'=\frac{1}{3}u^{-\frac{2}{3}}\cdot u',\quad u'=3bx^2$$

$$y'=\frac{1}{\cancel{3}u^{\frac{2}{3}}}\cdot\cancel{3}bx^2$$

$$y'=\frac{bx^2}{\sqrt[3]{(a+bx^3)^2}}$$

d) $f(y)=(2a+3by)^2$

Rj. $f'(y)=12ab+18b^2y$

e) $y=\sqrt{ctg\,x}-\sqrt{ctg\,\alpha}$

Rj. $y=\sqrt{u}-\sqrt{ctg\,\alpha}$, $u=ctg\,x$

$$y'=\frac{1}{2\sqrt{u}}\cdot u',\quad u'=-\frac{1}{\sin^2x}$$

$$y'=\frac{-1}{2\sin^2x\sqrt{ctg\,x}}$$

f) $y=2x+5\cos^3x$

Rj. $y'=2+15\cos^2x\cdot(-\sin x)$

$$y'=2-15\cos^2x\sin x$$

g) $f(x)=-\dfrac{1}{6(1-3\cos x)^2}$

Rj. $y'=\dfrac{\sin x}{(1-3\cos x)^3}$

Naći izvode sljedećih f-ja:

$y=x^4(a-2x^3)^2$

Rj.

$$y'=4x^3(a-2x^3)^2+x^4\cdot 2(a-2x^3)\cdot\overset{(-2)(3)}{(-6)}x^2$$

$$y'=4x^3(a-2x^3)\cdot[\underbrace{a-2x^3+x\cdot(-1)\cdot 3x^2}_{a-2x^3-3x^3}]$$

$$y'=4x^3(a-2x^3)(a-5x^3)$$

$y=tg^2\,5x$

Rj. $y'=2\,tg5x\cdot(tg5x)'$

$$y'=2\,tg5x\cdot\frac{1}{\cos^2x}\cdot(5x)'$$

$$y'=\frac{10\,tg5x}{\cos^2x}$$

$y=(a+x)\sqrt{a-x}$

Rj.

$$y'=1\cdot\sqrt{a-x}+(a+x)\frac{1}{2\sqrt{a-x}}\cdot(-1)$$

$$y'=\sqrt{a-x}-\frac{a+x}{2\sqrt{a-x}}=\frac{2(a-x)-(a+x)}{2\sqrt{a-x}}$$

$$y'=\frac{a-3x}{2\sqrt{a-x}}$$

$y=\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}$

Rj.

$$y'=\frac{1}{2\sqrt{\cos x}}\cdot(\cos x)'\cdot a^{\sqrt{\cos x}}+\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}\ln a\cdot(\sqrt{\cos x})'$$

$$y'=-\frac{\sin x}{2\sqrt{\cos x}}\cdot a^{\sqrt{\cos x}}+\ln a\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}\cdot\frac{1}{2\sqrt{\cos x}}\cdot(\cos x)'$$

$$y'=-\frac{\sin x}{2\sqrt{\cos x}}a^{\sqrt{\cos x}}-\frac{\ln a\cdot\sin x\cdot\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}$$

$$y'=-\frac{\sin x\,a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}\left[1+\ln a\cdot\sqrt{\cos x}\right]$$

$$y'=-\frac{\sin x\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x}\cdot\sqrt{\cos x}}\left[1+\ln a\cdot\sqrt{\cos x}\right]$$

$$y'=-\frac{1}{2}tg\,x\cdot y\cdot\left[1+\ln a\sqrt{\cos x}\right]$$

$z=\sqrt[3]{y+\sqrt{y}}$

Rj.

$$(\sqrt[3]{x})'=(x^{\frac{1}{3}})'=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}}$$

$$z'=\frac{1}{3\sqrt[3]{(y+\sqrt{y})^2}}\cdot(y+\sqrt{y})'$$

$$z'=\frac{1}{3\sqrt[3]{(y+\sqrt{y})^2}}\cdot\left(1+\frac{1}{2\sqrt{y}}\right)$$

$$z'=\frac{1}{3\sqrt[3]{(y+\sqrt{y})^2}}\cdot\frac{2\sqrt{y}+1}{2\sqrt{y}}$$

$$z'=\frac{2\sqrt{y}+1}{6\sqrt{y}\sqrt[3]{(y+\sqrt{y})^2}}$$

$y=3^{ctg\frac{1}{x}}$ Rj. $y'=\dfrac{3^{ctg\frac{1}{x}}\cdot\ln 3}{(x\sin\frac{1}{x})^2}$

$y=\ln(x+\sqrt{a^2+x^2})$ Rj. $y'=\dfrac{1}{\sqrt{a^2+x^2}}$

$y=\ln\frac{(x-2)^5}{(x+1)^3}$

Rj.

$$y=\ln(x-2)^5-\ln(x+1)^3$$

$$y'=\frac{1}{(x-2)^5}\cdot\left((x-2)^5\right)'-\frac{1}{(x+1)^3}\cdot\left[(x+1)^3\right]'$$

$$y'=\frac{5(x-2)^4}{(x-2)^5}-\frac{3(x+1)^2}{(x+1)^3}\ \ldots$$

y mogu napisati i kao

$$y=5\ln(x-2)-3\ln(x+1)$$

$$y'=5\cdot\frac{1}{x-2}-3\cdot\frac{1}{x+1}$$

$$y'=\frac{5(x+1)-3(x-2)}{(x-2)(x+1)}$$

$$y'=\frac{2x+11}{x^2-x-2}$$

$y=\ln\ln(3-2x^3)$

Rj. $y'=\frac{1}{\ln(3-2x^3)}\cdot\left(\ln(3-2x^3)\right)'$

$$y'=\frac{1}{\ln(3-2x^3)}\cdot\frac{1}{3-2x^3}\cdot(3-2x^3)'$$

$$y'=\frac{-6x^2}{(3-2x^3)\ln(3-2x^3)}$$

$y=\ln\frac{(x-1)^3(x-2)}{x-3}$

Rj. $y'=\frac{3x^2-16x+19}{(x-1)(x-2)(x-3)}$

$f(x)=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$

Rj. $y'=\frac{\sqrt{1+x^2}}{x}$

$y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$

Rj. prvo pojednostavimo izraz

$$\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\cdot\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}+x}=$$

$$=\frac{(\sqrt{x^2+a^2}+x)^2}{x^2+a^2-x^2}=\frac{(\sqrt{x^2+a^2}+x)^2}{a^2}$$

$$y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}=2\ln\frac{\sqrt{x^2+a^2}+x}{a^2}$$

$$y'=2\cdot\frac{1}{\frac{\sqrt{x^2+a^2}+x}{a^2}}\cdot\left(\frac{\sqrt{x^2+a^2}+x}{a^2}\right)'$$

$$y'=\frac{2\cancel{a^2}}{\sqrt{x^2+a^2}+x}\cdot\frac{1}{\cancel{a^2}}\cdot\left[\frac{1}{\cancel{2}\sqrt{x^2+a^2}}\cdot\overbrace{(x^2+a^2)'}^{\cancel{2}x}+1\right]$$

$$y'=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$y'=\frac{2}{\sqrt{x^2+a^2}}$$

$y=\operatorname{arctg}\ln x$

Rj. $y'=\frac{1}{1+\ln^2x}\cdot(\ln x)'$

$$y'=\frac{1}{x(1+\ln^2x)}$$

Izvodi f-ja koje nisu eksplicitno zadane

$y=f(x)$ je eksplicitni oblik f-je. Pored eksplicitnog oblika postoje: $\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$ parametarski oblik f-je

i $F(x,y)=0$ implicitan oblik f-je

1. Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana parametarski $\begin{cases}x=a\cos t\\ y=a\sin t\end{cases}$.

Rj. $\frac{dx}{dt}=-a\sin t$ $\quad\frac{dy}{dt}=a\cos t$

$$\frac{dy}{dx}=\frac{a\cos t}{-a\sin t}=-\operatorname{ctg}t$$

tj. $y'=-\operatorname{ctg}t$

2. Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana $\begin{cases}x=\sqrt{t}\\ y=\sqrt[3]{t}\end{cases}$.

Rj. $\frac{dx}{dt}=\frac{1}{2\sqrt{t}}$, $\frac{dy}{dt}=\frac{1}{3}t^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{t^2}}$

$$\frac{dy}{dx}=\frac{\frac{1}{3\sqrt[3]{t^2}}}{\frac{1}{2\sqrt{t}}}=\frac{2\sqrt{t}}{3\sqrt[3]{t^2}}=\frac{2}{3}\sqrt[6]{\frac{t^3}{t^4}}=\frac{2}{3\sqrt[6]{t}}$$

tj. $y'=\frac{2}{3\sqrt[6]{t}}$

3. Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana par. $\begin{cases}x=a\cos^3t\\ y=b\sin^3t\end{cases}$

Rj. $y'=-\frac{b}{a}\operatorname{tg}t$

4. Izračunati izvod y'_x ako je f-ja zadana implic. $x^3+y^3-3axy=0$.

Rj. $x^3+y^3-3axy=0$

$$3x^2+3y^2\cdot y'-3ay-3axy'=0$$

$$(3y^2-3ax)y'=3ay-3x^2\quad /:3$$

$$y'=\frac{ay-x^2}{y^2-ax}$$

5. Izračunati izvod y'_x ako je f-ja zadana implicitno $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Rj. $\frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\cdot 2y\cdot y'=0$

$$\frac{2y}{b^2}y'=-\frac{2x}{a^2}\quad /:2$$

$$y'=-\frac{xb^2}{ya^2}$$

6. Izračunati izvod y'_x ako je f-ja zadana implicitno

$\sqrt{x^2+y^2}=c\cdot\operatorname{arctg}\frac{y}{x}$. Rj. $y'=\frac{cy+x\sqrt{x^2+y^2}}{cx-y\sqrt{x^2+y^2}}$

Logaritamski izvod

Logaritamskim izvodom f-je $y=f(x)$ nazivamo izvodom logaritma te f-je tj. $(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}$.

1. Naći izvod složene eksplicitno zadane f-je $y=u^v$ ako je $u=\varphi(x)$ i $v=\psi(x)$.

Rj. $y=u^v \quad /\ln$

$\ln y = \ln u^v$

$\ln y = v\ln u \quad /'$

$\frac{1}{y}\cdot y' = v'\ln u + v\cdot\frac{1}{u}\cdot u' \quad /\cdot y$

$y' = y\left(v'\ln u + \frac{v}{u}u'\right)$

2. Izračunati y' ako je $y=(\sin x)^x$.

Rj. $y=(\sin x)^x \quad /\ln$

$\ln y = \ln(\sin x)^x$

$\ln y = x\ln\sin x \quad /'$

$\frac{1}{y}\cdot y' = \ln\sin x + x\,\frac{1}{\sin x}\cdot\overbrace{(\sin x)'}^{\cos x}$

$y' = y\left(\ln\sin x + x\cdot\frac{\cos x}{\sin x}\right)$

$y' = (\sin x)^x\left(\ln\sin x + x\,ctg\,x\right)$

3. Izračunati y' ako je $y=\sqrt[3]{x^2}\cdot\frac{1-x}{1+x^2}\cdot\sin^3 x\cdot\cos^2 x$.

Rj. $\ln y = \ln\sqrt[3]{x^2} + \ln\frac{1-x}{1+x^2} + \ln\sin^3 x + \ln\cos^2 x$

$\ln y = \frac{2}{3}\ln x + \ln\frac{1-x}{1+x^2} + \ln\sin^3 x + \ln\cos^2 x \quad /'$

$\left(\frac{1-x}{1+x^2}\right)' = \frac{(-1)(1+x^2)-(1-x)\cdot 2x}{(1+x^2)^2} = \frac{-1-x^2-2x+2x^2}{(1+x^2)^2} = \frac{x^2-2x-1}{(1+x^2)^2}$

$\frac{1}{y}\cdot y' = \frac{2}{3}\cdot\frac{1}{x} + \frac{\cancel{1+x^2}}{1-x}\cdot\frac{x^2-2x-1}{(1+x^2)^{\cancel{2}}} + \frac{3\cancel{\sin^2 x}}{\sin^{\cancel{3}} x}\cdot\cos x + \frac{2\cancel{\cos x}}{\cos^{\cancel{2}} x}\cdot(-\sin x)$

$y' = y\left(\frac{2}{3x} + \frac{x^2-2x-1}{(1-x)(1+x^2)} + 3\,ctg\,x - 2\,tg\,x\right)$

4. $y=x^x$, Rj. $y'=x^x(1+\ln x)$

5. $y=x^{x^2}$, Rj. $y'=x^{x^2+1}(1+2\ln x)$

6. $y=\sqrt[x]{x}$, Rj. $y'=\sqrt[x]{x}\;\frac{1-\ln x}{x^2}$

Primjena izvoda u geometriji

Ako je data kriva $y=f(x)$ i ako je $M(x_1,y_1)$ data tačka tada je $y-y_1=f'(x_1)(x-x_1)$ jednačina tangente u tački M.

$x-x_1+f'(x_1)(y-y_1)=0$ ili $y-y_1=\frac{-1}{f'(x_1)}(x-x_1)$ je jednačina normale na krivu tački $M(x_1,y_1)$

Ako su $y_1=k_1x+n_1$ i $y_2=k_2x+n_2$ dvije date prave tada je

$tg\,\varphi=\frac{k_2-k_1}{1+k_1\cdot k_2}$ tangens ugla između dvije prave

Pod uglom između dvije krive $y=f_1(x)$ i $y=f_2(x)$ u njihovoj presječnoj tački podrazumjevamo ugao φ između njihovih zajednički tangenti u presječnoj tački $N(x_1,y_1)$

$$tg\,\varphi = \frac{f_2'(x_1)-f_1'(x_1)}{1+f_1'(x_1)\cdot f_2'(x_1)}$$

1. Naći jednačinu tangente na krivu $y=2x^2-4x-6$ u tački $M(\frac{3}{2},\frac{-15}{2})$ i nacrtati sliku.

Rj. $y=2x^2-4x-6$

nacrtajmo ovu krivu

nule $y=0$

$2x^2-4x-6=0$

$2(x^2-2x-3)=0$

$2(x+1)(x-3)=0$

$x_1=3 \Rightarrow y=0$

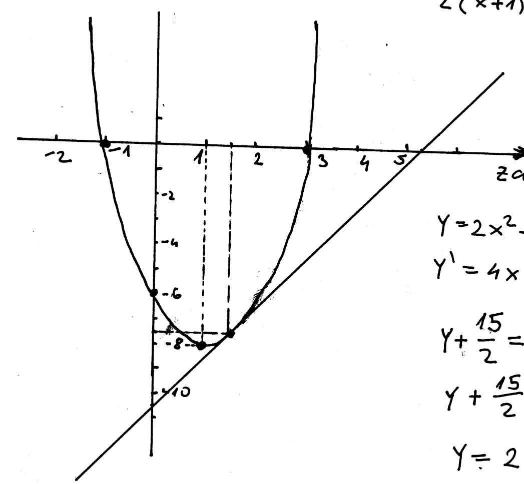
$x_2=-1 \Rightarrow y=0$

tjeme parabole $T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a}=\frac{4}{4}=1$

$-\frac{D}{4a}=-\frac{16+48}{8}=-\frac{64}{8}=-8$

za $x=0 \Rightarrow y=-6$



$y=2x^2-4x-6$

$y'=4x-4$

$M\in f(x)$

$y'(\frac{3}{2})=4\cdot\frac{3}{2}-4=6-4=2$

$y+\frac{15}{2}=2\left(x-\frac{3}{2}\right)$

$y+\frac{15}{2}=2x-3$

$y=2x-\frac{21}{2}$ jednačina tangente

(2.) Napišite jednačinu tangente i normale na krivu $y=x^3+2x^2-4x-3$ u tački $(-2,5)$.

Rj. $y'=3x^2+4x-4$

$y'(-2)=12-8-4=0$

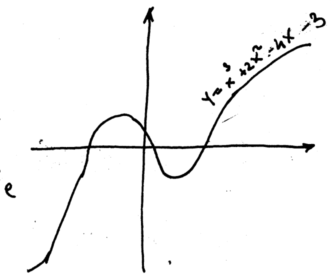
$y-y_0=f'(x_0)(x-x_0)$

$y-5=0(x+2)$

$y-5=0$ jednačina tangente

$x-x_0+y_0'(y-y_0)=0$ jedn. norm.

$x+2=0$ jedn. normale



(3.) Nađi jednačinu tangente i normale na krivu $y=\sqrt[3]{x-1}$ u tački $(1,0)$. Rj. $x-1=0$, $y=0$

(4.) Odrediti ugao pod kojim se sijeku krive $y=x^2$ i $x=y^2$!

Rj. Prvo nađimo tačke presjeka krivih.

$$\begin{aligned} y&=x^2\\ x&=y^2 \end{aligned}$$

$y=y^4$

$y-y^4=0$

$y^4-y=0$

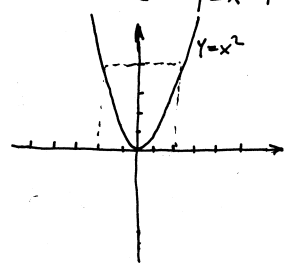
$y(y^3-1)=0$

$y(y-1)(y^2+y+1)=0$

$y_1=0$ ili $y_2=1$

$y_1=0 \Rightarrow x_1=0$

$y_2=1 \Rightarrow x_2=1$



Postoje dvije tačke presjeka $(0,0)$ i $(1,1)$

f_1: $y=x^2$

$y'=2x$

$f_1'(0)=0$

$f_1'(1)=2$

f_2: $x=y^2$

$1=2yy'$

$y'=\frac{1}{2y}$

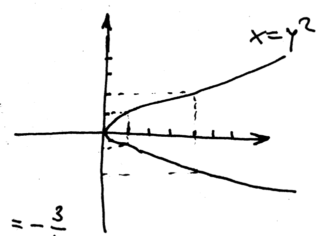
$f_2'(0)$ nije def.

$f_2'(1)=\frac{1}{2}$

$$tg\varphi=\frac{f_2'(x_0)-f_1'(x_0)}{1-f_1'(x_0)\cdot f_2'(x_0)}$$

$$tg\varphi=\frac{\frac{1}{2}-2}{1+\frac{1}{2}\cdot 2}=\frac{-\frac{3}{2}}{2}=-\frac{3}{4}$$

$\varphi=arc\,tg(-\frac{3}{4})$ ugao pod kojim se sijeku date krive u tački $(1;1)$.



(5.) Nađi ugao pod kojim se sijeku parabole $y=(x-2)^2$ i $y=-4+6x-x^2$.

Rj. $\varphi=40^\circ 36'$

Izvodi višeg reda

$y=f(x)$ — data f-ja

$y'=f'(x)$ prvi izvod

$y''=(f'(x))'=f''(x)$ drugi izvod

$y'''=[f''(x)]'=f'''(x)$ treći izvod

$\vdots$

$y^{(n)}=[y^{(n-1)}]'=f^{(n)}(x)$ n-ti izvod f-je $y=f(x)$

(1.) Nađi y''' f-je $y=xe^x$

Rj. $y=xe^x$

$y'=e^x+xe^x=(x+1)e^x$

$y''=e^x+(x+1)e^x=(x+2)e^x$

$y'''=e^x+(x+2)e^x=(x+3)e^x$

(2.) Nađi $y^{(5)}$ f-je $y=2x^3+3x^2-4x+5$

Rj. $y'=6x^2+6x-4$

$y''=12x+6$

$y'''=12$

$y^{(4)}=0$

$y^{(5)}=0$

(3.) Nađi y'' f-je $y=ln\,\frac{x^2+3}{x^2+1}$.

Rj.
$$y'=\frac{1}{\frac{x^2+3}{x^2+1}}\cdot\left(\frac{x^2+3}{x^2+1}\right)'=\frac{x^2+1}{x^2+3}\cdot\frac{2x\cdot(x^2+1)-(x^2+3)\cdot 2x}{(x^2+1)^2}$$

$$y'=\frac{2x^3+2x-2x^3-6x}{(x^2+3)(x^2+1)}=\frac{-4x}{(x^2+3)(x^2+1)}=\frac{-4x}{x^4+4x^2+3}$$

$$y''=\frac{(-4)(x^4+4x^2+3)-(-4x)(4x^3+8x)}{(x^2+3)^2(x^2+1)^2}=\frac{-4x^4-16x^2-12+16x^4+32x^2}{(x^2+3)^2(x^2+1)^2}=\frac{12x^4+16x^2-12}{(x^2+3)^2(x^2+1)^2}$$

$$y''=\frac{4(3x^4+4x^2-3)}{(x^2+3)^2(x^2+1)^2}$$

4. Naći y'' f-je $y=(x-1)e^{-\frac{1}{x+1}}$

Rj. $y'=\left((x-1)e^{-\frac{1}{x+1}}\right)' = e^{-\frac{1}{x+1}} + (x-1)e^{-\frac{1}{x+1}}\cdot\left(-\frac{1}{x+1}\right)' =$

$$= e^{-\frac{1}{x+1}} + (x-1)\cdot\frac{1}{(x+1)^2}e^{-\frac{1}{x+1}} = \left(1+\frac{x-1}{(x+1)^2}\right)e^{-\frac{1}{x+1}}$$

$$\left(-\frac{1}{(x+1)}\right)' = \left[-(x+1)^{-1}\right]' = (x+1)^{-2} \qquad y' = \frac{(x+1)^2+x-1}{(x+1)^2}e^{-\frac{1}{x+1}}$$

$$y' = \frac{x^2+2x\cancel{+1}+x\cancel{-1}}{(x+1)^2}e^{-\frac{1}{x+1}} = \frac{x(x+3)}{(x+1)^2}e^{-\frac{1}{x+1}} = \frac{(x^2+3x)e^{-\frac{1}{x+1}}}{x^2+2x+1}$$

$$y'' = \left[\frac{x(x+3)e^{-\frac{1}{x+1}}}{(x+1)^2}\right]' = \frac{\left[(2x+3)e^{-\frac{1}{x+1}}+(x^2+3x)e^{-\frac{1}{x+1}}\cdot\frac{1}{(x+1)^2}\right]\cdot(x+1)^2-(x^2+3x)e^{-\frac{1}{x+1}}\cdot 2(x+1)}{(x+1)^4}$$

$$y'' = \frac{[(2x+3)\overbrace{(x+1)^2}^{x^2+2x+1}+x^2+3x-2\overbrace{(x^2+3x)(x+1)}^{x^3+x^2+3x^2+3x}]}{(x+1)^4}e^{-\frac{1}{x+1}}$$

$$y'' = \frac{\cancel{2x^3}+4x^2+2x+3x^2+6x+3+x^2+3x\cancel{-2x^3}-8x^2-6x}{(x+1)^4}e^{-\frac{1}{x+1}}$$

$$y'' = \frac{5x+3}{(x+1)^4}e^{-\frac{1}{x+1}}$$

5. Naći y'' f-ja:

a) $y=\dfrac{x^3}{x^2-2x-8}$ Rj. $y''=\dfrac{24x(x^2+4x+16)}{(x^2-3x-8)^3}$

b) $y=\dfrac{16}{x^2\cdot(x-4)}$ Rj. $y''=\dfrac{64(3x^2-16x+24)}{x^4(x-4)^3}$

c) $y=(2x-1)e^{-\frac{x}{x-1}}$ Rj. $y''=\dfrac{e^{-\frac{x}{x-1}}}{(x-1)^4}$

L'Hospital-Bernoullijevo pravilo

Ako su obe f-je $f(x)$ i $g(x)$ beskonačno male ili beskonačno velike kad $x\to a$ tj. ako razlomak $\frac{f(x)}{g(x)}$ predstavlja u tački $x=a$ neodređen oblik tipa $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ tada je $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$.

Neodređene limese koji su oblika $0\cdot\infty$, $\infty-\infty$, 1^∞, 0^0, ∞^0 skoro uvijek možemo svesti na neki od oblika $\frac{0}{0}$ ili $\frac{\infty}{\infty}$ i onda ih naći pomoću Lopitalovog pravila.

Izračunati:

1. $\lim\limits_{x\to 0}\frac{\ln x}{ctg\, x}\left(\frac{-\infty}{\infty}\right) \overset{LoP}{=} \lim\limits_{x\to 0}\frac{(\ln x)'}{(ctg\, x)'} = \lim\limits_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = -\lim\limits_{x\to 0}\frac{\sin^2 x}{x} =$

$= -\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\sin x = -1\cdot 0 = 0$

2. $\lim\limits_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 1}\frac{3x^2-4x-1}{3x^2-7} = \frac{-2}{-4} = \frac{1}{2}$

3. $\lim\limits_{x\to 0}\frac{x\cos x-\sin x}{x^3}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 0}\frac{\overbrace{\cos x + x(-\sin x) - \cos x}^{-x\sin x}}{3x^2}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 0}\frac{-\sin x+(-x)\cos x}{6x}$

$\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 0}\frac{-\cos x-\cos x-x(-\sin x)}{6} = \frac{-2}{6} = -\frac{1}{3}$

4. $\lim\limits_{x\to 1}\frac{1-x}{1-\sin\frac{\pi x}{2}}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 1}\frac{-1}{-\cos\frac{\pi x}{2}\cdot\frac{\pi}{2}} = \frac{-1}{-0} = +\infty$

5. $\lim\limits_{x\to 0}\frac{tg\, x-\sin x}{x-\sin x}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 0}\frac{\frac{1}{\cos^2 x}-\cos x}{1-\cos x} = \lim\limits_{x\to 0}\frac{\frac{1-\cos^3 x}{\cos^2 x}}{1-\cos x}$

$= \lim\limits_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2 x)}{\cos^2 x(1-\cos x)} = 3$

6. $\lim\limits_{x\to 0}\frac{\sin 5x}{x}\left(\frac{0}{0}\right) \overset{LoP}{=} \lim\limits_{x\to 0}\frac{\cos 5x\cdot 5}{1} = 5$

7. $\lim\limits_{x\to\infty}\frac{e^x}{x^5}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim\limits_{x\to\infty}\frac{e^x}{5x^4}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim\limits_{x\to\infty}\frac{e^x}{20x^3}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}\ldots=\frac{\infty}{120}=\infty$

8. $\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3\sqrt[3]{x^2}}}=\lim\limits_{x\to\infty}\frac{3\sqrt[3]{x^2}}{x}=3\lim\limits_{x\to\infty}\frac{x^{\frac{2}{3}}:x}{x\;:x}=3\lim\limits_{x\to\infty}x^{-\frac{1}{3}}=0$

9. $\lim\limits_{x\to0}\frac{\ln(\sin mx)}{\ln\sin x}$ Rj. 1

10. $\lim\limits_{x\to1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)(\infty-\infty)=\lim\limits_{x\to1}\frac{\ln x-(x-1)}{(x-1)\ln x}\left(\frac{0}{0}\right)\overset{L.P.}{=}$

$=\lim\limits_{x\to1}\frac{\frac{1}{x}-1}{\ln x+(x-1)\frac{1}{x}}=\lim\limits_{x\to1}\frac{\frac{1}{x}-1}{\ln x-\frac{1}{x}+1}\left(\frac{0}{0}\right)\overset{L.P.}{=}\lim\limits_{x\to1}\frac{-\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=-\frac{1}{2}$

11. $\lim\limits_{x\to0}(1-\cos x)\,\mathrm{ctg}\,x\,(0\cdot\infty)=\lim\limits_{x\to0}\frac{(1-\cos x)\cos x}{\sin x}\left(\frac{0}{0}\right)=$

$=\lim\limits_{x\to0}\frac{1-\cos x}{\sin x}\cdot\lim\limits_{x\to0}\cos x\overset{L.P.}{=}\lim\limits_{x\to0}\frac{\sin x}{\cos x}\cdot\lim\limits_{x\to0}\cos x=0\cdot1=0$

12. $\lim\limits_{x\to\infty}\left[x\cdot\left(e^{-\frac{2}{x}}-1\right)\right](\infty\cdot0)=\lim\limits_{x\to\infty}\frac{e^{-\frac{2}{x}}-1}{\frac{1}{x}}\left(\frac{0}{0}\right)\overset{L.P.}{=}\lim\limits_{x\to\infty}\frac{e^{-\frac{2}{x}}\cdot(-2)\cdot(-1)\cdot x^{-2}}{(-1)\cdot x^{-2}}$

$=e^0\cdot(-2)=-2$

13. $\lim\limits_{x\to\infty}x\cdot\sin\frac{a}{x}$ Rj. a

14. $\lim\limits_{x\to1}x^{\frac{1}{1-x}}\left(1^\infty\right)=\lim\limits_{x\to1}e^{\ln x^{\frac{1}{1-x}}}=\lim\limits_{x\to1}e^{\frac{1}{1-x}\cdot\ln x}=e^{\lim\limits_{x\to1}\frac{\ln x}{1-x}}\left(\frac{0}{0}\right)=$

$\overset{L.P.}{=}e^{\lim\limits_{x\to1}\frac{\frac{1}{x}}{-1}}=e^{-1}=\frac{1}{e}$

15. $\lim\limits_{x\to0}(\mathrm{ctg}\,x)^{\frac{1}{\ln x}}\left(\infty^0\right)=\lim\limits_{x\to0}e^{\ln(\mathrm{ctg}\,x)^{\frac{1}{\ln x}}}=e^{\lim\limits_{x\to0}\frac{\ln(\mathrm{ctg}\,x)}{\ln x}}\left(\frac{\infty}{\infty}\right)$

$\overset{L.P.}{=}e^{\lim\limits_{x\to0}\frac{\frac{1}{\mathrm{ctg}\,x}\cdot\frac{-1}{\sin^2x}}{\frac{1}{x}}}=e^{\lim\limits_{x\to0}\frac{-\frac{\sin x}{\cos x}\cdot\frac{1}{\sin^2x}}{\frac{1}{x}}}=e^{\lim\limits_{x\to0}\frac{-x}{\sin x\cos x}}\left(\frac{0}{0}\right)\overset{L.P.}{=}e^{\lim\limits_{x\to0}\frac{-1}{\cos^2x-\sin^2x}}$

$=e^{-1}=\frac{1}{e}$

16. $\lim\limits_{x\to0}x^{\sin x}$ Rj. 1

17. $\lim\limits_{x\to\infty}\left[(x-1)e^{\frac{-1}{x+1}}-x\right]$ Rj. -2

\# Ako je $h(x)=\frac{1}{\sin x}-\frac{1}{x}$ izračunati $\lim\limits_{x\to0}h'(x)$.

Rj. $h(x)=\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}$

$h'(x)=\left(\frac{1}{\sin x}\right)'-\left(\frac{1}{x}\right)'=\left(\sin^{-1}x\right)'-\left(x^{-1}\right)'=(-1)\sin^{-2}x\cdot\cos x-(-1)x^{-2}$

$h'(x)=\frac{-\cos x}{\sin^2x}+\frac{1}{x^2}=\frac{1}{x^2}-\frac{\cos x}{\sin^2x}=\frac{\sin^2x-x^2\cos x}{x^2\sin^2x}$

$\lim\limits_{x\to0}h'(x)=\lim\limits_{x\to0}\frac{\sin^2x-x^2\cos x}{x^2\sin^2x}\left(=\frac{0}{0}\right)\overset{L.P.}{=}$

$\overset{L.P.}{=}\lim\limits_{x\to0}\frac{\overbrace{2\sin x\cos x}^{\sin 2x}-\left(2x\cos x+x^2(-\sin x)\right)}{2x\sin^2x+x^2\underbrace{2\sin x\cos x}_{\sin2x}}=\lim\limits_{x\to0}\frac{\sin2x-2x\cos x+x^2\sin x}{2x\sin^2x+x^2\sin2x}$

$\left(=\frac{0}{0}\right)\overset{L.P.}{=}\lim\limits_{x\to0}\frac{\cos2x\cdot2-2(\cos x+x(-\sin x))+(2x\sin x+x^2\cos x)}{2(\sin^2x+x\underbrace{2\sin x\cos x}_{\sin2x})+2x\sin2x+x^2\cos2x\cdot2}=$

$=\lim\limits_{x\to0}\frac{2\cos2x-2\cos x+2x\sin x+2x\sin x+x^2\cos x}{2\sin^2x+2x\sin2x+2x\sin2x+2x^2\cos2x}=$

$=\lim\limits_{x\to0}\frac{2\cos2x-2\cos x+x^2\cos x+4x\sin x}{2\sin^2x+2x^2\cos2x+4x\sin2x}$

$=\lim\limits_{x\to0}\frac{2(-\sin2x)\cdot2-2(-\sin x)+(2x\cos x+x^2(-\sin x))+4\sin x+4x\cos x}{2\cdot\underbrace{2\sin x\cos x}_{\sin2x}+2(2x\cos2x+x^2(-\sin2x)\cdot2)+4\sin2x+4x\cos2x\cdot2}$

$=\lim\limits_{x\to0}\frac{-4\sin2x+6\sin x+6x\cos x-x^2\sin x}{6\sin2x+12x\cos2x-4x^2\sin2x}\left(=\frac{0}{0}\right)$

$=\lim\limits_{x\to0}\frac{-4\cos2x\cdot2+6\cos x+6(\cos x+x(-\sin x))\cdot(2x\sin x+x^2\cos x)}{6\cos2x\cdot2+12(\cos2x+x(-\sin2x)\cdot2)-4(2x\sin2x+x^2\cos2x\cdot2)}=$

$=\frac{-8+6+6}{12+12}=\frac{4}{24}=\frac{1}{6}$

Prema tome $\lim\limits_{x\to0}h'(x)=\frac{1}{6}$

Ispitivanje f-je

Ispitati f-ju znači odrediti

a) oblast definisanosti

b) parnost (neparnost) i periodičnost

c) nule, presjek grafa sa Y-osom, znak f-je

d) ponašanje na krajevima intervala definisanosti i asimptote

e) rast i opadanje f-je (intervale u kojima f-ja raste ili opada)

f) ekstreme f-je (minimum i maksimum ako ih ima)

g) prevojne tačke i intervale konveksnosti i konkavnosti

h) na osnovu svega ovoga nacrtati graf

Definiciono područje obilježavat ćemo sa $\mathcal{D}$ i to je skup svih onih vrijednosti u kojima je f-ja definisana (ima konačnu ili beskonačnu vrijednost).

1. Odrediti definiciono područje sljedećih f-ja:

a) $y=\frac{1}{x}$, Rj. $\mathcal{D}: \mathbb{R}\setminus\{0\}$ ili $\mathcal{D}: x\in(-\infty,0)\cup(0,+\infty)$

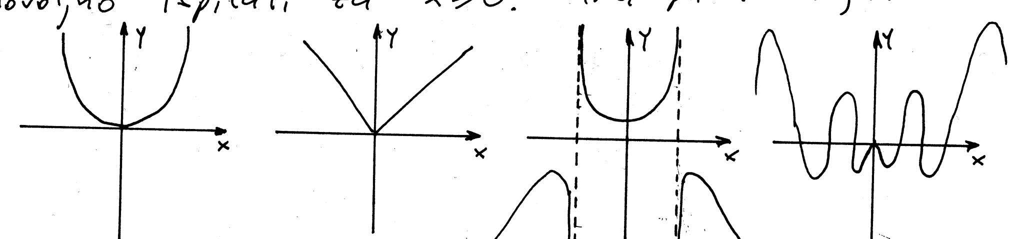
b) $y=\sqrt{x}$, Rj. $\mathcal{D}: x\in\mathbb{R}_0^+$ ili $\mathcal{D}: x\in[0,+\infty)$ ili $\mathcal{D}: x\geq 0$

c) $y=\log x$, Rj. $\mathcal{D}: x\in\mathbb{R}^+$ ili $\mathcal{D}: x\in(0,+\infty)$ ili $\mathcal{D}: x>0$

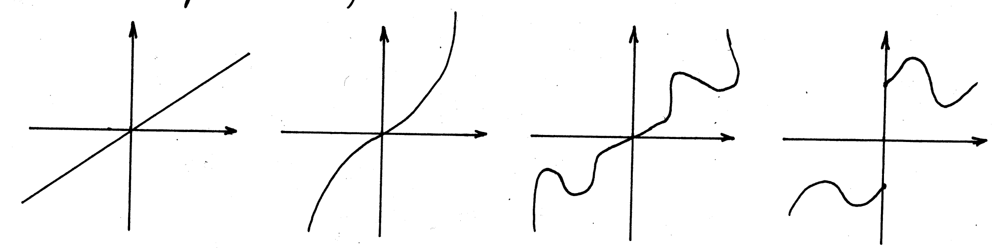
d) $y=\frac{1}{\sqrt{x}}$, Rj. $\mathcal{D}: x\in\mathbb{R}^+$ ili $\mathcal{D}: x\in(0,+\infty)$ ili $\mathcal{D}: x>0$

e) $y=\frac{\log x}{x-2}$, Rj. $x>0$, $x-2\neq 0$, $\mathcal{D}: x\in\mathbb{R}^+\setminus\{2\}$ ili $\mathcal{D}: x\in(0,2)\cup(2,+\infty)$

F-ja je parna ako je $\forall(x\in\mathcal{D})$ $f(-x)=f(x)$. Grafik parne f-je je simetričan u odnosu na Y-osu i f-ju je dovoljno ispitati za $x\geq 0$. Grafici parnih f-ja:



Ako je $\forall(x\in\mathcal{D})$ $f(-x)=-f(x)$ f-ja $f(x)$ je neparna f-ja. Grafik neparne f-je je simetričan u odnosu na koordinatni početak $(0,0)$ pa je f-ju dovoljno ispitati za $x\geq 0$. Grafici neparnih f-ja:



2. Odrediti parnost i neparnost sljedećih f-ja

a) $y=\frac{x^3}{x^2-4}$ Rj. $f(-x)=\frac{(-x)^3}{(-x)^2-4}=\frac{-x^3}{x^2-4}=-\frac{x^3}{x^2-4}=-f(x)$

f-ja je neparna

b) $y=\frac{x^2+1}{\sqrt{x^2-1}}$ Rj. $f(-x)=\frac{(-x)^2+1}{\sqrt{(-x)^2-1}}=\frac{x^2+1}{\sqrt{x^2-1}}=f(x)$ f-ja $f(x)$ je parna

c) $y=\frac{(x+1)^3}{(x-1)^2}$ Rj. Parnost i neparnost ima smisla ispitati samo ako je $\mathcal{D}$ simetrično. U našem slučaju $\mathcal{D}: (-\infty,1)\cup(1,+\infty)$ nije simetrično pa f-ja nije ni parna ni neparna.

II način: $f(-x)=\frac{(-x+1)^3}{(-x-1)^2}$ $\Rightarrow$ f-ja nije ni parna ni neparna

---

Neka je data f-ja $y=f(x)$.

Ako je za svako $x\in(a,b)$ $y'(x)<0$ tada f-ja y opada ($\searrow$) na (a,b)

Ako je za svako $x\in(a,b)$ $y'(x)>0$ tada f-ja y $\nearrow$ (raste) na (a,b)

Rješenjem jednačine $y'=0$ dobijamo stacionarne tačke $x_1, x_2, \dots, x_n$ koje konkurišu za ekstrem. Stacionarne tačke $x_1, x_2, \dots, x_n$ mogu ali i ne moraju da budu tačke u kojima f-ja poprima ekstrem. Da li je stacionarna tačka x_1 ekstrem možemo zaključiti na dva načina:

I način: Na osnovu tabele rasta i opadanja, $\nearrow \overset{x_1}{} \searrow$ MAX, $\searrow \overset{x_1}{} \nearrow$ MIN

II način: x_1 je stacionarna tačka

Ako je $y''(x_1)<0 \Rightarrow (x_1, f(x_1))$ je tačka u kojoj f-ja y ima maksimalnu vrijednost

Ako je $y''(x_1)>0 \Rightarrow (x_1, f(x_1))$ je tačka u kojoj f-ja y ima minimalnu vrijednost

3. Nađi ekstreme i intervale rasta i opadanja sljedećih f-ja:

a) $y=\frac{x^3}{x^2-4}$ Rj. $\mathcal{D}: x\in(-\infty,-2)\cup(-2,2)\cup(2,+\infty)$

$$y'=\left(\frac{x^3}{x^2-4}\right)'=\frac{3x^2(x^2-4)-x^3\cdot 2x}{(x^2-4)^2}=\frac{x^2(3x^2-12-2x^2)}{(x^2-4)^2}=\frac{x^2(x^2-12)}{(x^2-4)^2}$$

$y''=0$ ako i samo ako $x^2=0$ ili $x^2-12=0$

$x=0$ ili $x_{1,2}=\pm\sqrt{12}$ tj. $x_{1,2}=\pm 2\sqrt{3}$

Stacionarne tačke su $x_1=0$, $x_2=-2\sqrt{3}$ i $x_3=2\sqrt{3}$.

prekidi f-je y + nule f-je y'

$-2\sqrt{2}$ -2 0 2 $2\sqrt{2}$

rast i opadanje

x	$(-\infty,-2\sqrt{3})$	$(-2\sqrt{3},-2)$	$(-2,0)$	$(0,2)$	$(2,2\sqrt{3})$	$(2\sqrt{3},+\infty)$
y'	+	−	−	−	−	+
y	↗	↘	↘	↘	↘	↗

MAX MIN

$$f(-2\sqrt{3})=\frac{(-2\sqrt{3})^3}{(-2\sqrt{3})^2-4}=\frac{-24\sqrt{3}}{8}=-3\sqrt{3}$$

$$f(2\sqrt{3})=\frac{24\sqrt{3}}{12-4}=3\sqrt{3}$$

$M(-2\sqrt{3},-3\sqrt{3})$ je tačka lokalnog maksimuma a tačka $N(2\sqrt{3},3\sqrt{3})$ je tačka lokalnog minimuma

b) $y=\frac{x^2+1}{\sqrt{x^2-1}}$ Rj. $\mathcal{D}: x\in(-\infty,-1)\cup(1,+\infty)$

$$y'=\frac{2x\sqrt{x^2-1}-(x^2+1)\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}}{x^2-1}=\frac{2x(x^2-1)-x(x^2+1)}{(x^2-1)\sqrt{x^2-1}}=\frac{x(2x^2-2-x^2-1)}{(x^2-1)\sqrt{x^2-1}}$$

$y'=\frac{x(x^2-3)}{(x^2-1)\sqrt{x^2-1}}$, $y'=0$ akko $x=0$ ili $x^2-3=0$

$x_{1,2}=\pm\sqrt{3}$

prekidi f-je y + nule f-je y'

Stacionarne tačke su $x_1=0$, $x_2=-\sqrt{3}$ ili $x_3=\sqrt{3}$.

$-\sqrt{3}$ -1 0 1 $\sqrt{3}$

x	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},-1)$	$(-1,0)$	$(0,1)$	$(1,\sqrt{3})$	$(\sqrt{3},+\infty)$
y'	−	+			−	+
y	↘	↗			↘	↗

rast i opadanje

MIN MIN

Tačke $M(-\sqrt{3},2\sqrt{2})$ i $N(\sqrt{3},2\sqrt{2})$ su tačke lokalnog minimuma.

$$f(-\sqrt{3})=\frac{3+1}{\sqrt{3-1}}=\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=2\sqrt{2},\quad f(\sqrt{3})=2\sqrt{2}$$

4. Ispitati i grafički predstaviti f-ju $y=\frac{x}{x-3}$.

Rj. definiciono područje

$x-3\neq 0$

$x\neq 3$ $\mathcal{D}: (-\infty,3)\cup(3,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna.

F-ja $f(x)$ je periodična sa periodom T ako $f(x+T)=f(x)$. Periodične su samo trigonometrijske f-je

F-ja nije periodična

nule, presjek sa y-osom, znak f-je

tačka oblika $(A,0)$ je nula f-je, a tačka oblika $(0,B)$ je tačka presjeka sa y-osom.

$f(x)=\frac{x}{x-3}$, $f(0)=\frac{0}{-3}=0$ $(0,0)$ je nula f-je i presjek sa y-osom

prekidi f-je i nule y

0 3

x	$(-\infty,0)$	$(0,3)$	$(3,+\infty)$
x	−	+	+
$x-3$	−	−	+
y	+	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

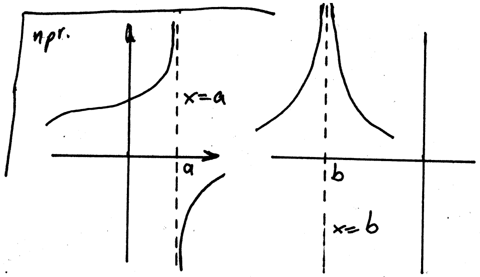
Neka je a tačka u kojoj f-ja nije definisana.

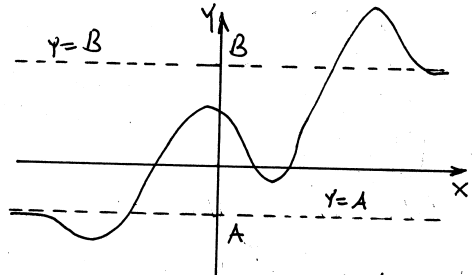
$\lim_{x\to a-0} f(x)=-\infty$ (ili $+\infty$) $\Rightarrow$ $x=a$ je vertikalna asimptota

$\lim_{x\to a+0} f(x)=+\infty$ (ili $-\infty$) $\Rightarrow$ $x=a$ je vertikalna asimptota

$\lim_{x\to\infty} f(x)=A$, $A\neq+\infty$ i $A\neq-\infty$ $\Rightarrow$ $y=A$ je horizontalna asimptota

$\lim_{x\to-\infty} f(x)=B$, $B\neq+\infty$ i $B\neq-\infty$ $\Rightarrow$ $y=B$ je horizontalna asimptota



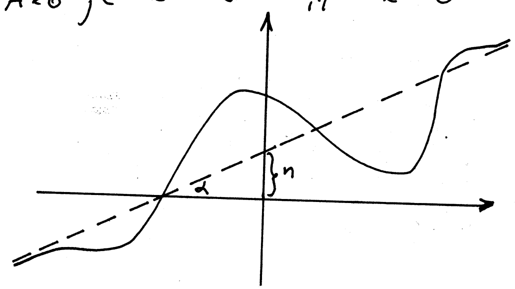


$x=a$ i $x=b$ su V.A. $y=A$ i $y=B$ su H.A.

Ako f-ja nema horizontalnu asimptotu onda tražimo kosu asimptotu u obliku $y=kx+n$.

$k=\lim_{x\to\infty}\frac{f(x)}{x}$, $n=\lim_{x\to\infty}[f(x)-kx]$

Ako je $k=\pm\infty$ ili $k=0$ f-ja nema kosu asimptotu.



U beskonačnosti f-ja ne dodiruje asimptotu ali je u "normalnom" položaju u nekoj tački može sijeći.

Za $x=3$ f-ja nije definisana

$\lim_{x\to 3-0}\frac{x}{x-3}=\frac{3-0}{3-0-3}=\frac{3-0}{-0}=-\infty$

$\Rightarrow x=3$ je V.A. (sa lijeve str.)

$\lim_{x\to 3+0}\frac{x}{x-3}=\frac{3+0}{3+0-3}=\frac{3+0}{+0}=+\infty$

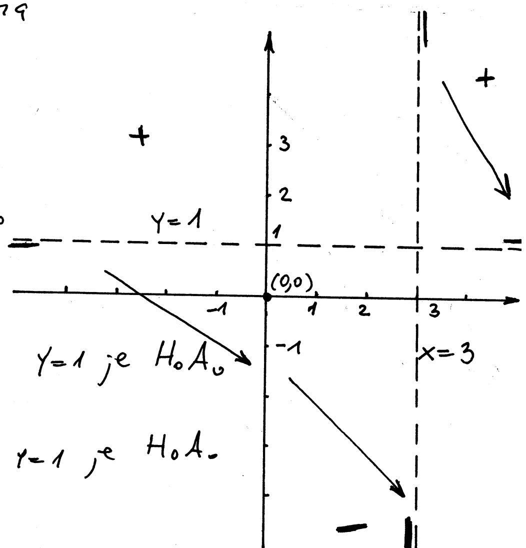
$\Rightarrow x=3$ je V.A. (sa desne str.)

$\lim_{x\to+\infty}\frac{x}{x-3}\,\frac{:x}{:x}=\lim_{x\to+\infty}\frac{1}{1-\frac{3}{x}}=1 \Rightarrow y=1$ je H.A.

$\lim_{x\to-\infty}\frac{x}{x-3}=\lim_{x\to-\infty}\frac{1}{1-\frac{3}{x}}=1 \Rightarrow y=1$ je H.A.

F-ja nema kosu asimptotu.

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je.



intervali rasta i opadanja

$y'=\left(\frac{x}{x-3}\right)'=\frac{1\cdot(x-3)-x\cdot 1}{x-3}=\frac{-3}{(x-3)^2}<0\quad \forall x\in D$

f-ja y ↘ za $\forall(x\in D)$

ekstremi f-je

$y'=0$, $y'=\frac{-3}{(x-3)^2}\neq 0\quad \forall x\in D \Rightarrow$ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

Konveksnost ($\cup$) i konkavnost ($\cap$) f-je određujemo na osnovu znaka f-je y''.
Ako je $\forall x\in(a,b)$ $y''(x)<0 \Rightarrow$ f-ja y je $\cap$ na (a,b)
Ako je $\forall x\in(a,b)$ $y''(x)>0 \Rightarrow$ f-ja y je $\cup$ na (a,b)
Za $y''=0$ dobijemo tačke $x_1, x_2, \dots, x_n$ koje konkurišu za prevojne tačke. Tačka x_1 je prevojna tačka ako u njoj f-ja y prelazi iz $\cup$ u $\cap$ i obrnuto

$y''=\left(\frac{-3}{(x-3)^2}\right)'=\left(-3(x-3)^{-2}\right)'=6(x-3)^{-3}\cdot 1=\frac{6}{(x-3)^3}\neq 0 \Rightarrow$ f-ja nema prevojnih tački

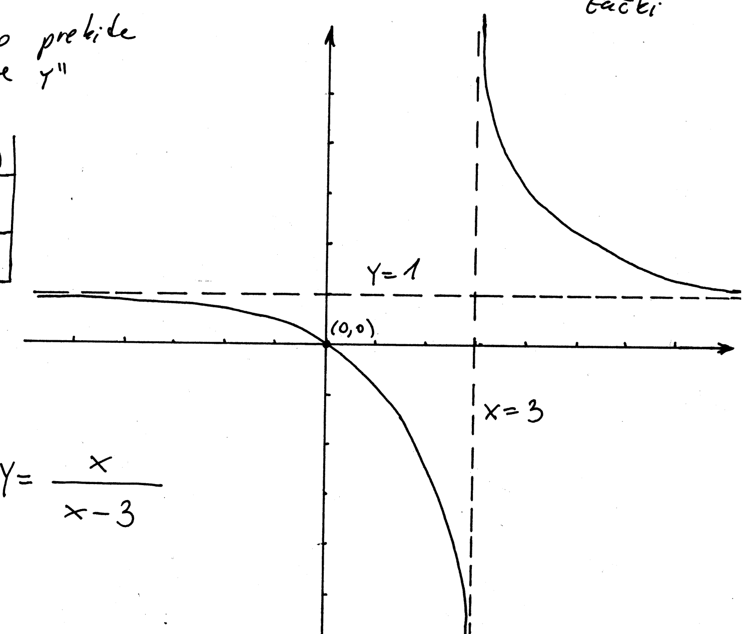
u tabelu stavljamo prekide f-je y + nule f-je y''

x	$(-\infty,3)$	$(3,+\infty)$
y''	−	+
y	$\cap$	$\cup$

konveksnost i konkavnost

grafik f-je

$y=\frac{x}{x-3}$



Ispitati f-ju i nacrtati joj grafik $y=\frac{3x}{1+x^3}$.

Rj. definiciono područje

$1+x^3\neq 0$

$x^3\neq -1$

$x\neq -1$

$D: x\in(-\infty,-1)\cup(-1,+\infty)$

parnost, neparnost, periodičnost

$f(-x)=\frac{3\cdot(-x)}{1+(-x^3)} = -\frac{3x}{1-x^3}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$

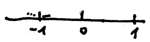
$\frac{3x}{1+x^3}=0$

$x=0$

$(0,0)$ je nula f-je i presjek sa y-osom

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
$3x$	$-$	$-$	$+$
$1+x^3$	$-$	$+$	$+$
y	$+$	$-$	$+$

znak f-je



ponašanje na krajevima intervala definisanosti i asimptote

za vrijednost $x=-1$ f-ja ima prekid

$\lim_{x\to-1-0} f(x)=\lim_{x\to-1-0}\frac{3x}{1+x^3}=\frac{3(-1-0)}{1+(-1-0)^3}=\frac{3(-1-0)}{1-1-0}=\frac{-3-0}{-0}=+\infty \Rightarrow x=-1$ je V.A.

$\lim_{x\to-1+0} f(x)=\lim_{x\to-1+0}\frac{3x}{1+x^3}=\frac{3(-1+0)}{1+(-1+0)^3}=\frac{-3+0}{1-1+0}=\frac{-3+0}{+0}=-\infty \Rightarrow x=-1$ je V.A.

$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{3x\ :x}{1+x^3\ :x}=\lim_{x\to-\infty}\frac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2}=0 \Rightarrow y=0$ je H.A.

$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2}=0 \Rightarrow y=0$ je H.A. f-ja nema K.A.

rast i opadanje

$y'=\left(\frac{3x}{1+x^3}\right)'=3\cdot\frac{1\cdot(1+x^3)-x\cdot 3x^2}{(1+x^3)^2}=3\frac{1+x^3-3x^3}{(1+x^3)^2}$

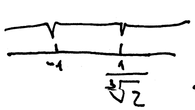
$y'=3\cdot\frac{1-2x^3}{(1+x^3)^2}$

$y'=0$ akko $1-2x^3=0$

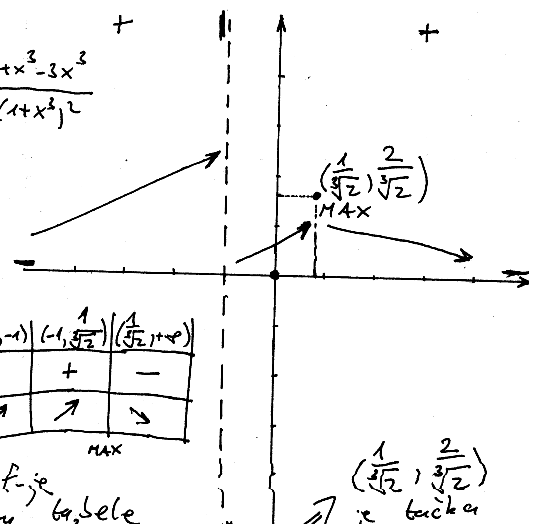
$2x^3=1$

$x^3=\frac{1}{2}$

$x=\frac{1}{\sqrt[3]{2}}\approx 0,8$



prekidi y + nule y'



x	$(-\infty,-1)$	$\left(-1,\frac{1}{\sqrt[3]{2}}\right)$	$\left(\frac{1}{\sqrt[3]{2}},+\infty\right)$
y'	$+$	$+$	$-$
y	↗	↗	↘
			MAX

ekstrem f-je

Na osnovu tabele

$f\left(\frac{1}{\sqrt[3]{2}}\right)=\frac{\frac{3}{\sqrt[3]{2}}}{1+\frac{1}{2}}=\frac{\frac{3}{\sqrt[3]{2}}}{\frac{3}{2}}=\frac{2}{\sqrt[3]{2}}\approx 1.6$

$\Rightarrow$ $\left(\frac{1}{\sqrt[3]{2}},\frac{2}{\sqrt[3]{2}}\right)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

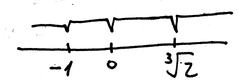
$y'=3\cdot\frac{1-2x^3}{(1+x^3)^2}$, $y''=3\cdot\frac{-6x^2\cdot(1+x^3)^{\cancel{2}}-(1-2x^3)\cdot 2\cancel{(1+x^3)}\cdot 3x^2}{(1+x^3)^3\cdot\cancel{(1+x^3)}}=$

$=3\cdot\frac{-6x^2-6x^5-6x^2+12x^5}{(1+x^3)^3}=3\cdot\frac{6x^5-12x^2}{(1+x^3)^3}$

$y''=18\cdot\frac{x^5-2x^2}{(1+x^3)^3}=\frac{18x^2(x^3-2)}{(1+x^3)^3}$

$y''=0$ akko $x=0$ ili $x^3-2=0$

$x_1=0$ $\quad x_2=\sqrt[3]{2}\approx 1,3$

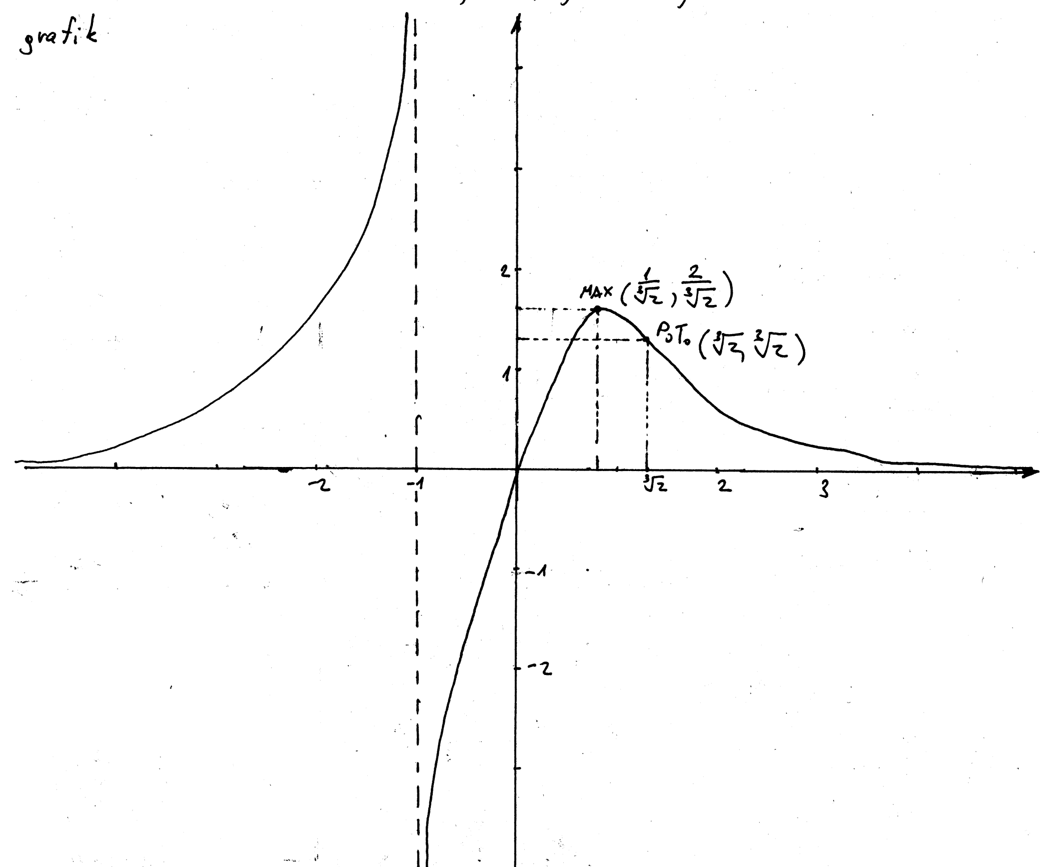


x	$(-\infty,-1)$	$(-1,0)$	$(0,\sqrt[3]{2})$	$(\sqrt[3]{2},+\infty)$
y''	$+$	$-$	$-$	$+$
y	$\cup$	$\cap$	$\cap$	$\cup$
				P.T.

$f(\sqrt[3]{2})=\frac{3\sqrt[3]{2}}{1+2}=\sqrt[3]{2}$

$(\sqrt[3]{2},\sqrt[3]{2})$ je prevojna tačka

grafik



Ispitati f-ju i nacrtati joj grafik $Y=\frac{(2x-1)^3}{(x+2)^2}$.

Rj.

definiciono područje

$D: x\in R\setminus\{-2\}$

parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$Y=0$ akko $(2x-1)^3=0$

$2x-1=0$

$x=\frac{1}{2}$

$(\frac{1}{2},0)$ je nula f-je

$f(0)=\frac{(-1)^3}{2^2}=-\frac{1}{4}$

$(0,-\frac{1}{4})$ je tačka presjeka sa y-osom

x	$(-\infty,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
$(2x-1)^3$	−	−	+
Y	−	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$$\lim_{x\to-2-0}f(x)=\lim_{x\to-2-0}\frac{(2x-1)^3}{(x+2)^2}=\frac{(2\cdot(-2-0)-1)^3}{(-2-0+2)^2}=\frac{(-5-0)^3}{+0}=-\infty \Rightarrow x=-2 \text{ je V.A.}$$

(sa lijeve strane)

$$\lim_{x\to-2+0}f(x)=\lim_{x\to-2+0}\frac{(2x-1)^3}{(x+2)^2}=\frac{(2\cdot(-2+0)-1)^3}{(-2+0+2)^2}=\frac{(-5-0)^3}{+0}=-\infty \Rightarrow x=-2 \text{ je V.A.}$$

(sa desne strane)

$$(2x-1)^3=(2x)^3+3\cdot(2x)^2\cdot(-1)+3\cdot2x\cdot(-1)^2+(-1)^3=8x^3-12x^2+6x-1$$

1
1 1
1 2 1
1 3 3 1

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{(2x-1)^3}{(x+2)^2}=\lim_{x\to-\infty}\frac{8x^3-12x^2+6x-1}{x^2+4x+2}\,{:x^3}=\lim_{x\to-\infty}\frac{8-\frac{12}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{\frac{1}{x}+\frac{4}{x^2}+\frac{2}{x^3}}=-\infty$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{8x^3-12x^2+6x-1}{x^2+4x+2}\,{:x^2}=\lim_{x\to+\infty}\frac{8x-12+\frac{6}{x}-\frac{1}{x^2}}{1+\frac{4}{x}+\frac{2}{x^2}}=+\infty$$

f-ja nema H.A.

kosa asimptota je oblika $y=kx+n$

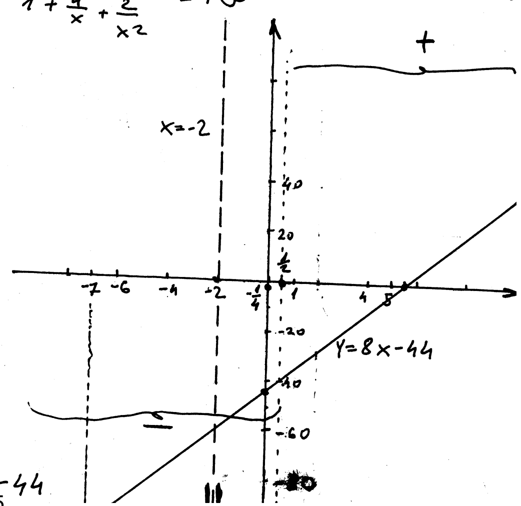
$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{8x^3-12x^2+6x-1}{x^3+4x^2+2x}\,{:x^3}=\lim_{x\to\infty}\frac{8-\frac{12}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{1+\frac{4}{x}+\frac{2}{x^2}}=8$$

$$n=\lim_{x\to\infty}[f(x)-k\cdot x]=\lim_{x\to\infty}\left(\frac{(2x-1)^3}{(x+2)^2}-8x\right)=$$

$$=\lim_{x\to\infty}\frac{8x^3-12x^2+6x-1-8x(x^2+4x+2)}{(x+2)^2}=$$

$$=\lim_{x\to\infty}\frac{\cancel{8x^3}-12x^2+6x-1-\cancel{8x^3}-32x^2-16x}{x^2+4x+4}$$

$$=\lim_{x\to\infty}\frac{-44x^2-10x-1}{x^2+4x+4}\,{:x^2}=\lim_{x\to\infty}\frac{-44-\frac{10}{x}-\frac{1}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}=-44$$



$Y=8x-44$ je K.A. (počinjemo sa skiciranjem grafa)

($Y=8x-44$, $Y=0 \Rightarrow 8x=44$, $x=\frac{44}{8}=\frac{11}{2}=5,5$ $\quad x=0 \Rightarrow Y=-44$)

rast i opadanje

$$Y'=\left(\frac{(2x-1)^3}{(x+2)^2}\right)'=\frac{3(2x-1)^2\cdot2(x+2)^{\cancel{2}\,1}-(2x-1)^3\,2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}}=\frac{2(2x-1)^2(3x+6-2x+1)}{(x+2)^3}=\frac{2(2x-1)^2(x+7)}{(x+2)^3}$$

$Y'=0$ akko $x=\frac{1}{2}$ i $x=-7$

-7 -2 $\frac{1}{2}$ ← nule Y' + prekidi od Y

x	$(-\infty,-7)$	$(-7,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
Y'	+	−	+	+
Y	↗	↘	↗	↗

max

rast i opadanje

$$f(-7)=\frac{(-15)^3}{(-5)^2}=\frac{-3375}{25}=-135$$

ekstremi f-je Na osnovu tabele rasta i opadanja $M(-7,-135)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(2\frac{(2x-1)^2(x+7)}{(x+2)^3}\right)'=2\cdot\frac{[2(2x-1)\cdot2\cdot(x+7)+(2x-1)^2](x+2)^{\cancel{3}\,1}-(2x-1)^2(x+7)3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}\,4}}=$$

$$=2\cdot\frac{[(2x-1)(4x+28+2x-1)](x+2)-3(2x-1)^2(x+7)}{(x+2)^4}=2\cdot\frac{(2x-1)[\overbrace{(6x+27)(x+2)}^{6x^2+12x+27x+54}-3\overbrace{(2x-1)(x+7)}^{2x^2+13x-7}]}{(x+2)^4}$$

$$Y''=2\cdot\frac{(2x-1)(\cancel{6x^2}+39x+54-\cancel{6x^2}-39x+21)}{(x+2)^4}=\frac{150(2x-1)}{(x+2)^4}$$

$Y''=0$ akko $x=\frac{1}{2}$

-2 $\frac{1}{2}$ ← prekidi Y + nule Y''

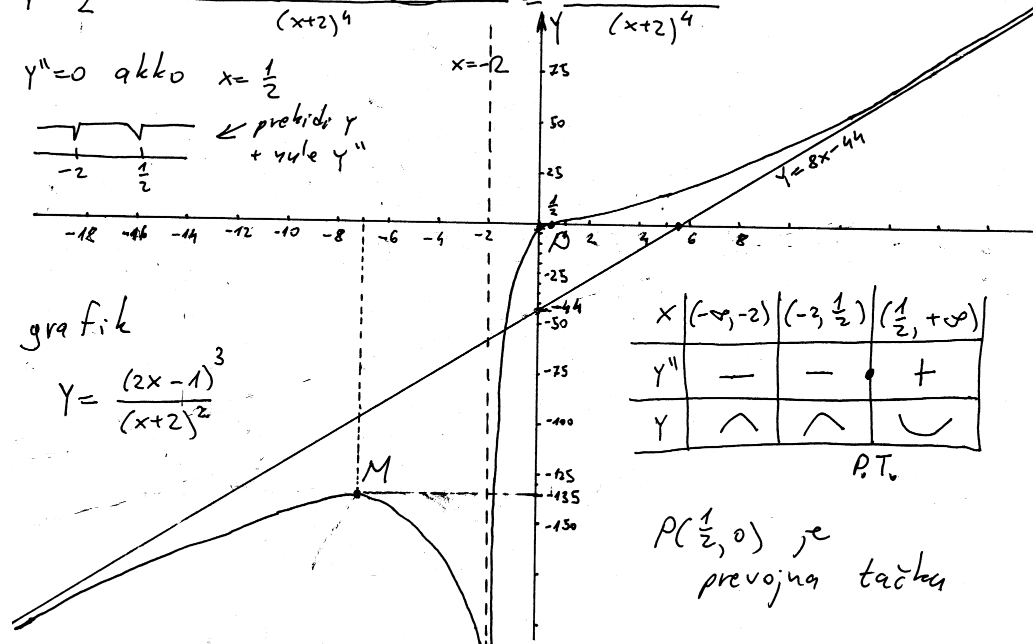
grafik

$$Y=\frac{(2x-1)^3}{(x+2)^2}$$

x	$(-\infty,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
Y''	−	−	+
Y	∩	∩	∪

P.T.

$P(\frac{1}{2},0)$ je prevojna tačka



(#) Ispitati i grafički predstaviti f-ju $Y=\frac{x^2+5x}{x^2+2x+1}$.

Rj.

definiciono područje

$x^2+2x+1\neq 0$ $\quad D: x\in R\setminus\{-1\}$

$D=4-4=0$

$(x+1)^2\neq 0$

$x\neq -1$

parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$

f-ja nije ni parna ni neparna

f-ja nije periodična

$Y=\frac{x(x+5)}{(x+1)^2}$ $\quad -5 \quad -1 \quad 0$

nule, presjek sa y-osom, znak f-je

$Y=0$ akko $x^2+5x=0$

$x(x+5)=0$

$x_1=0$ ili $x_2=-5$

$(0,0)$ i $(-5,0)$ su nule f-je

$(0,0)$ je tačka presjeka sa Y-osom.

x	$(-\infty,-5)$	$(-5,-1)$	$(-1,0)$	$(0,+\infty)$
x	$-$	$-$	$-$	$+$
$x+5$	$-$	$+$	$+$	$+$
Y	$+$	$-$	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ f-ja ima prekid.

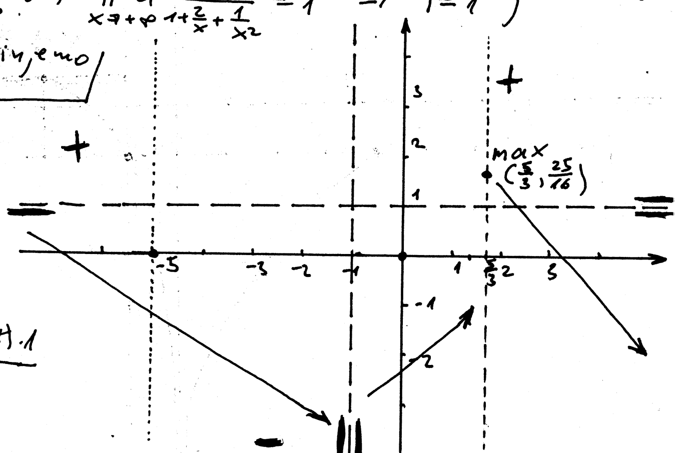
$$\lim_{x\to-1-0} f(x)=\lim_{x\to-1-0}\frac{x(x+5)}{(x+1)^2}=\frac{(-1-0)(-1-0+5)}{(-1-0+1)^2}=\frac{(-1-0)(4-0)}{+0}=-\infty \Rightarrow x=-1 \text{ je } V_0A_0$$

$$\lim_{x\to-1+0} f(x)=\lim_{x\to-1+0}\frac{x(x+5)}{(x+1)^2}=\frac{(-1+0)(-1+0+5)}{(-1+0+1)^2}=\frac{(-1+0)(4+0)}{+0}=-\infty \Rightarrow x=-1 \text{ je } V_0A_0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x^2+5x \;/:x^2}{x^2+2x+1 \;/:x^2}=\lim_{x\to-\infty}\frac{1+\frac{5}{x}}{1+\frac{2}{x}+\frac{1}{x^2}}=1 \Rightarrow Y=1 \text{ je } H_0A_0$$

isto vrijedi i za $\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{1+\frac{5}{x}}{1+\frac{2}{x}+\frac{1}{x^2}}=1 \Rightarrow Y=1$ je H_0A_0

nakon ovog koraka počinjemo skicirati graf



f-ja nema K_0A_0

rast i opadanje

$$Y'=\left(\frac{x^2+5x}{(x+1)^2}\right)'=$$

$$=\frac{(2x+5)(x+1)^{\cancel{2}\,1}-(x^2+5x)\,2\cancel{(x+1)}\cdot 1}{(x+1)^{\cancel{4}\,3}}$$

$$=\frac{\cancel{2x^2}+5x+2x+5-\cancel{2x^2}-10x}{(x+1)^3}$$

$Y'=\frac{-3x+5}{(x+1)^3}$ $\quad -1 \quad \frac{5}{3}$ $\quad \leftarrow$ nule y' + prekidi Y

x	$(-\infty,-1)$	$(-1,\frac{5}{3})$	$(\frac{5}{3},+\infty)$
Y'	$-$	$+$	$-$
Y	$\searrow$	$\nearrow$	$\searrow$

max $\quad$ rast i opadanje

$Y'=0$ akko $-3x+5=0$

$-3x=-5$

$x=\frac{5}{3}\approx 1,6667$

$Y'(-2)=\frac{11}{-1}<0$

ekstremi f-je

na osnovu tabele rasta i opadanja f-ja ima maksimum za $x=\frac{5}{3}$

$$f\left(\frac{5}{3}\right)=\frac{\frac{25}{9}+5\cdot\frac{5}{3}}{\left(\frac{5}{3}+1\right)^2}=\frac{\frac{25+25\cdot 3}{9}}{\left(\frac{8}{3}\right)^2}=\frac{\frac{100}{\cancel{9}}\;:2}{\frac{64}{\cancel{9}}\;:2}=\frac{50}{32}=\frac{25}{16}\approx 1,5625$$

$M\left(\frac{5}{3},\frac{25}{16}\right)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(\frac{-3x+5}{(x+1)^3}\right)'=\frac{-3(x+1)^{\cancel{3}\,1}-(-3x+5)3\cancel{(x+1)^2}\cdot 1}{(x+1)^4\cdot\cancel{(x+1)^2}}=\frac{-3x-3+9x-15}{(x+1)^4}=\frac{6x-18}{(x+1)^4}$$

$Y''=6\cdot\frac{x-3}{(x+1)^4}$, $\quad Y''=0$ akko $x=3$

$-1 \quad 3$ $\quad \leftarrow$ prekidi Y + nule Y''

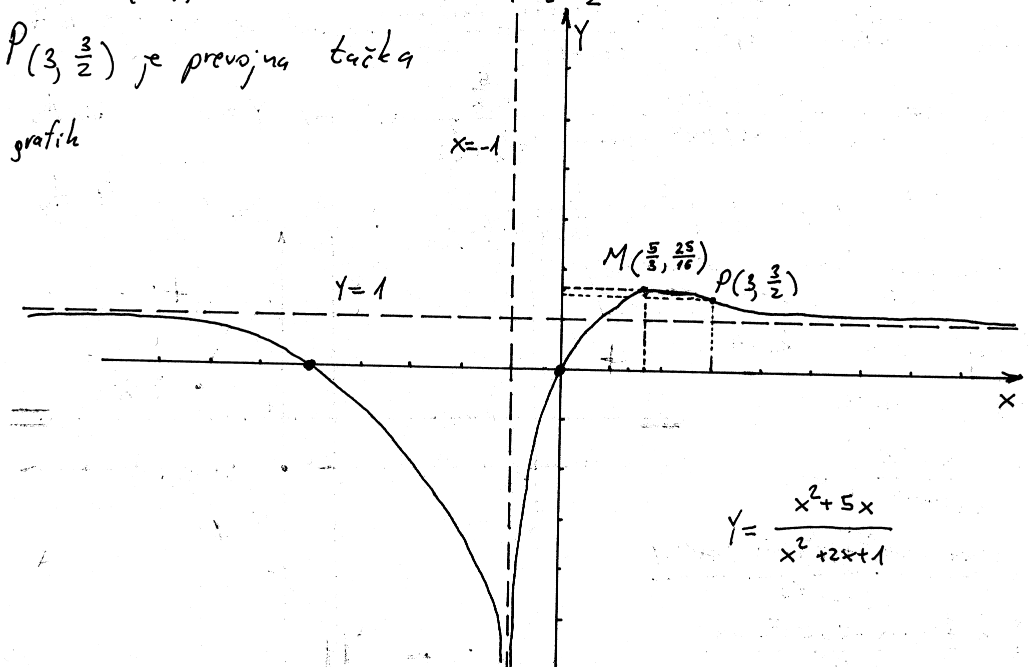
x	$(-\infty,-1)$	$(-1,3)$	$(3,+\infty)$
Y''	$-$	$-$	$+$
Y	$\frown$	$\frown$	$\smile$

P_0T_0

$$f(3)=\frac{3^2+5\cdot 3}{(3+1)^2}=\frac{9+15}{16}=\frac{24\;:4}{16\;:4}=\frac{6\;:2}{4\;:2}=\frac{3}{2}=1,5$$

$P\left(3,\frac{3}{2}\right)$ je prevojna tačka

grafik



\# Odrediti parametre a i b tako da f-ja $y=\frac{x}{x^2+ax+b}$ ima ekstrem u tački $T(2,\frac{1}{7})$. Zatim ispitati tako dobijenu f-ju i nacrtati joj grafik.

Rj.

$f(2)=\frac{1}{7}$

$\frac{2}{4+2a+b}=\frac{1}{7}$

$4+2a+b=14$

$2a+b=10$

Kandidat za ekstreme su stacionarne tačke

$y'=\frac{x^2+ax+b-x(2x+a)}{(x^2+ax+b)^2}=\frac{x^2+\cancel{ax}+b-2x^2-\cancel{ax}}{(x^2+ax+b)^2}$

$y'=\frac{-x^2+b}{(x^2+ax+b)^2}$

Potreban uslov da f-ja y ima ekstrem u tački $T(2,\frac{1}{7})$ je $y'(2)=0$.

$-4+b=0$

$b=4$

$2a+4=10$

$2a=6$

$a=3$

$y=\frac{x}{x^2+3x+4}$

definiciono područje

$x^2+3x+4\neq 0$

$D=9-16<0$

$a>0 \quad x^2+3x+4>0 \quad \forall x\in\mathbb{R}$

$\mathcal{D}: x\in\mathbb{R}$

parnost, neparnost, periodičnost

$f(-x)=\frac{-x}{x^2-3x+4}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$f(x)=0$ akko $x=0$

$(0,0)$ je nula f-je i presjek sa y-osom

x	$(-\infty,0)$	$(0,+\infty)$
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekida $\Rightarrow$ f-ja nema V.A.

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{x}{x^2+3x+4}\begin{matrix}:x\\:x\end{matrix}=\lim\limits_{x\to\infty}\frac{1}{x+3+\underbrace{\frac{4}{x}}_{0}}=\frac{1}{\infty}=0$

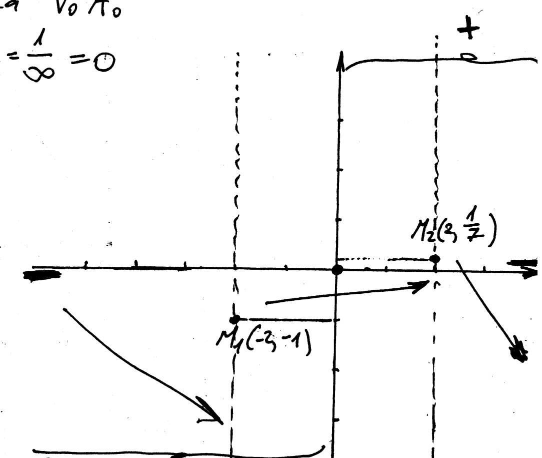
$\Rightarrow y=0$ je H.A.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\frac{1}{x+3+\frac{4}{x}}=\frac{1}{-\infty}=0$

$\Rightarrow y=0$ je H.A.

F-ja nema K.A.

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$y'=\frac{-x^2+b}{(x^2+ax+b)^2} \Rightarrow y'=\frac{4-x^2}{(x^2+3x+4)^2}$

$y'=0$ akko $4-x^2=0$

$x_1=-2,\ x_2=2$

$\frac{-2}{8-6}$

$\frac{2}{8+6}=\frac{2}{14}$

x	$(-\infty,-2)$	$(-2,2)$	$(2,+\infty)$
y'	−	+	−
y	↘	↗	↘

rast i opadanje

min $f(-2)=-1$, max $f(2)=\frac{1}{7}$

ekstremi f-je

Na osnovu tabele $M_1(-2,-1)$ je tačka min a $M_2(2,\frac{1}{7})$ je max.

prevojne tačke i intervali konv. i konk.

$y''=\left(\frac{4-x^2}{(x^2+3x+4)^2}\right)'=$

$=\frac{-2x(x^2+3x+4)^{\cancel{2}}-(4-x^2)2\cancel{(x^2+3x+4)}\cdot(2x+3)}{(x^2+3x+4)^{\cancel{4}\,3}}=\frac{-2\left[\underline{x^3}+\cancel{3x^2}+4x+8x+12\underline{-2x^3}-\cancel{3x^2}\right]}{(x^2+3x+4)^3}$

$y''=-2\cdot\frac{-x^3+12x+12}{(x^2+3x+4)^3}=2\,\frac{x^3-12x-12}{(x^2+3x+4)^3}$

$y''=0$ akko $x^3-12x-12=0$

$x_1\approx 3{,}88 \quad x_2\approx -1{,}11$

$x_3\approx -2{,}77$

(vrijednosti x_1, x_2 i x_3 su nađene pomoću digitrona koji ima opciju da nađe nule polinoma)

$-2{,}77 \quad -1{,}11 \quad 3{,}88$

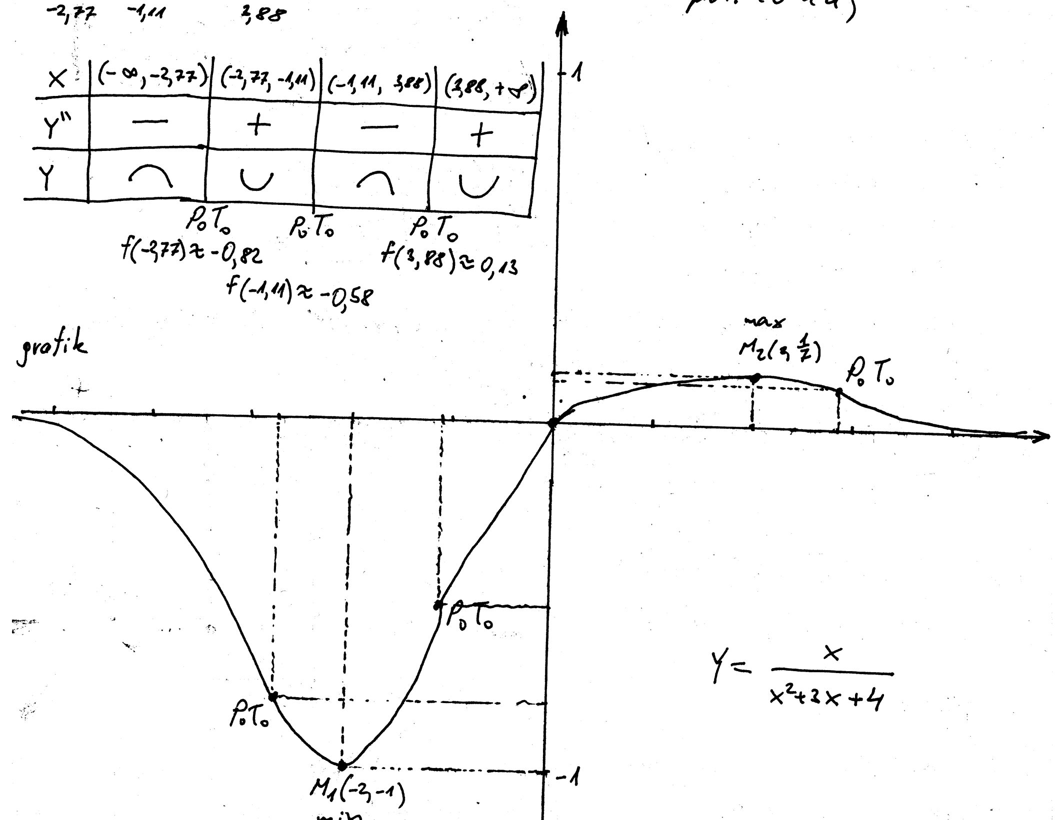
x	$(-\infty,-2{,}77)$	$(-2{,}77,-1{,}11)$	$(-1{,}11,\ 3{,}88)$	$(3{,}88,+\infty)$
y''	−	+	−	+
y	∩	∪	∩	∪

P.T. $f(-2{,}77)\approx-0{,}82$

P.T. $f(-1{,}11)\approx-0{,}58$

P.T. $f(3{,}88)\approx 0{,}13$

grafik



Ispitati i grafički predstaviti f-ju $y = x e^{\frac{1}{x}}$.

Rj. definiciono područje

$x \neq 0$, $D: x \in \mathbb{R} \setminus \{0\}$

parnost, neparnost, periodičnost

$f(-x) = -x e^{\frac{1}{-x}} = -x e^{-\frac{1}{x}}$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom, znak f-je

$x e^{\frac{1}{x}} = 0$

$x = 0$ ili $e^{\frac{1}{x}} = 0$

nije definisano $\quad e^{x} \neq 0 \ \forall x \in \mathbb{R}$

f-ja nema nulu

$f(0)$ nije definisano

f-ja ne siječe Y-osu

$e^{\frac{1}{x}} > 0 \ \forall x \in D$

x	$(-\infty, 0)$	$(0, +\infty)$
Y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$x = 0$ f-ja ima prekid $\qquad \left(-\frac{1}{x}\right)' = \left(-x^{-1}\right)'$

$$\lim_{x\to -0} f(x) = \lim_{x\to -0} x e^{\frac{1}{x}} = (-0)\cdot e^{\frac{1}{-0}} = (-0)\cdot e^{-\infty} = \frac{-0}{e^{\infty}} = \frac{-0}{\infty} = 0$$

$$\lim_{x\to +0} f(x) = \lim_{x\to +0} x e^{\frac{1}{x}} \ (= 0\cdot\infty) = \lim_{x\to +0} \frac{x}{e^{-\frac{1}{x}}} \left(= \frac{0}{0}\right) \overset{L.P.}{=} \lim_{x\to +0} \frac{1}{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}} = \lim_{x\to +0} \frac{x^2}{e^{-\frac{1}{x}}} =$$

pokušat ćemo na drugi način:

$$\lim_{x\to +0} f(x) = \lim_{x\to +0} x e^{\frac{1}{x}} \ (= 0\cdot\infty) = \lim_{x\to 0+} \frac{e^{\frac{1}{x}}}{x^{-1}} \left(= \frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to 0+} \frac{e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'} = e^{\frac{1}{0+}} = \infty$$

$\Rightarrow x = 0$ je V. A.

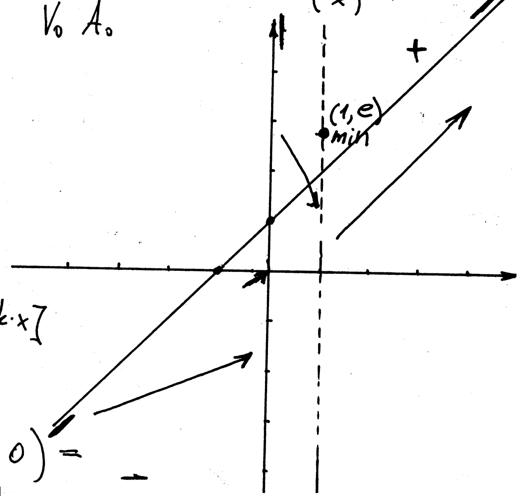
$$\left.\begin{aligned} &\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} x e^{\frac{1}{x}} = -\infty\cdot 1 = -\infty \\ &\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} x e^{\frac{1}{x}} = +\infty\cdot 1 = \infty \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ f-ja nema H. A.

$y = kx + n$, $\quad k = \lim_{x\to\infty} \frac{f(x)}{x}$, $\quad n = \lim_{x\to\infty} [f(x) - k\cdot x]$

$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^{0} = 1$

$n = \lim_{x\to\infty} [x e^{\frac{1}{x}} - x] = \lim_{x\to\infty} x(e^{\frac{1}{x}} - 1) \ (= \infty\cdot 0) =$



$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \left(= \frac{0}{0}\right) \overset{L.P.}{=} \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\cdot\cancel{\left(\frac{1}{x}\right)'}}{\cancel{\left(\frac{1}{x}\right)'}} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^{0} = 1$$

$y = x + 1$ je K. A.

rast i opadanje

$$y' = \left(x e^{\frac{1}{x}}\right)' = e^{\frac{1}{x}} + x\cdot e^{\frac{1}{x}}\cdot\left(x^{-1}\right)' = e^{\frac{1}{x}} + x e^{\frac{1}{x}}\cdot\left(-x^{-2}\right) = e^{\frac{1}{x}}\left(1 + x\cdot\left(-\frac{1}{x^2}\right)\right)$$

$y' = e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)$

$y' = 0$ akko $1 - \frac{1}{x} = 0$

$x = 1$

0 1

prekidi y + nule y'

x	$(-\infty, 0)$	$(0, 1)$	$(1, +\infty)$
y'	$+$	$-$	$+$
y	↗	↘	↗

MIN

rast i opadanje

ekstremi f-je

na osnovu tabele rasta i opadanja f-ja ima minimum u tački $(1, f(1))$. $\quad f(1) = 1\cdot e^{\frac{1}{1}} = e \quad f_{min}(1) = e \quad (1, e)$

$e \approx 2{,}71$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)\right)' = e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\left(1 - \frac{1}{x}\right) + e^{\frac{1}{x}}\cdot\left(-\left(-\frac{1}{x^2}\right)\right)$$

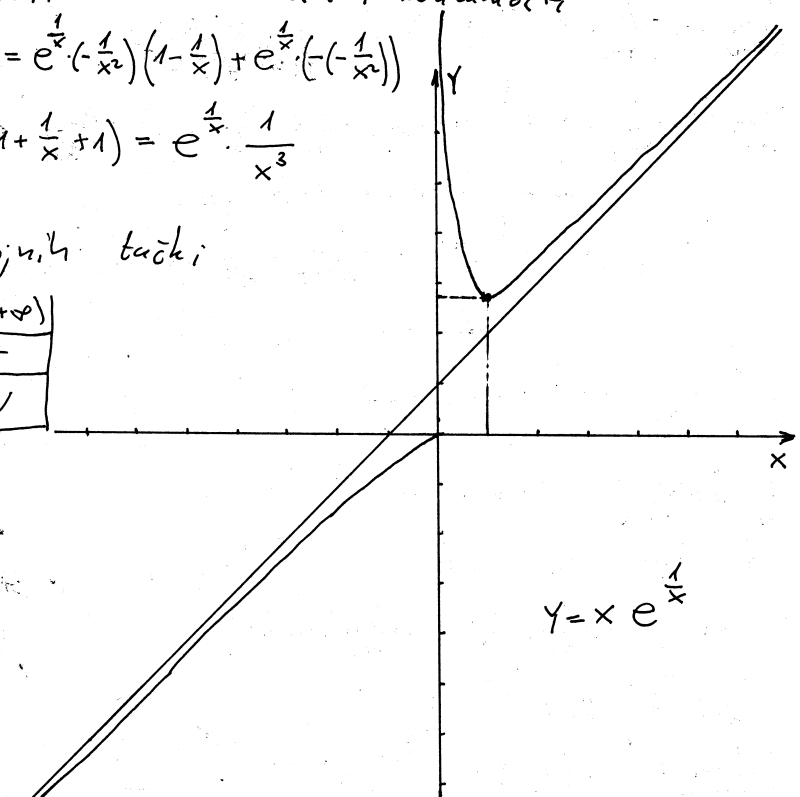
$$= e^{\frac{1}{x}}\cdot\frac{1}{x^2}\left(-1 + \frac{1}{x} + 1\right) = e^{\frac{1}{x}}\cdot\frac{1}{x^3}$$

$y'' \neq 0 \ \forall x \in D$

nema prevojnih tački

x	$(-\infty, 0)$	$(0, +\infty)$
y''	$-$	$+$
y	$\cap$	$\cup$

grafik



Ispitati f-ju i nacrtati joj grafik $y = x^3 e^{-\frac{x^2}{6}}$.

Rj.

definiciono područje

$\mathcal{D}: x \in \mathbb{R}$

parnost, neparnost, periodičnost

$y(-x) = (-x)^3 e^{-\frac{(-x)^2}{6}} = -x^3 e^{-\frac{x^2}{6}}$

f-ja je neparna (simetrična u odnosu na koordinatni početak). Dovoljno ju je ispitati za $x > 0$. F-ja nije periodična

nule, presjek sa y-osom, znak f-je

$x^3 \underbrace{e^{-\frac{x^2}{6}}}_{>0\ \forall x} = 0$

$x = 0$

(0,0) je nula f-je i presjek sa y-osom

x	$(-\infty, 0)$	$(0, +\infty)$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekid $\Rightarrow$ nema V.A.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 e^{-\frac{x^2}{6}} = \lim_{x\to+\infty} \frac{x^3}{e^{\frac{x^2}{6}}} \left(\frac{+\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to+\infty} \frac{3x^2}{e^{\frac{x^2}{6}}\cdot\frac{1}{6}2x} =$$

$$= \lim_{x\to+\infty} \frac{9x}{e^{\frac{x^2}{6}}} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to\infty} \frac{9}{e^{\frac{x^2}{6}}\cdot\frac{1}{6}2x} = \lim_{x\to\infty} \frac{27}{x e^{\frac{x^2}{6}}} = 0$$

$\Rightarrow$ $x=0$ je H.A., F-ja nema K.A.

rast i opadanje

$$y' = 3x^2 e^{-\frac{x^2}{6}} + x^3 \cdot e^{-\frac{x^2}{6}} \cdot \left(-\frac{1}{6}\right)\cdot 2x$$

$$= 3x^2 e^{-\frac{x^2}{6}} - \frac{1}{3}x^4 e^{-\frac{x^2}{6}}$$

$$= x^2 e^{-\frac{x^2}{6}}\left(3 - \frac{1}{3}x^2\right) = x^2 e^{-\frac{x^2}{6}}\left(\frac{9-x^2}{3}\right)$$

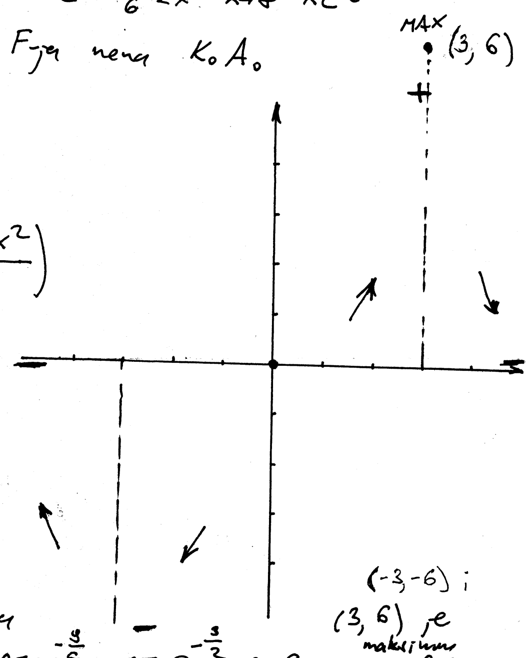
$y' = 0 \Leftrightarrow x_1 = 0,\ x_2 = -3,\ x_3 = 3$

← prekidi y + nule y'

x	$(0, 3)$	$(3, +\infty)$
y'	$+$	$-$
y	↗	↘

MAX

rast i opadanje



ekstremi f-je

Iz tabele rasta i opadanja vidimo da f-ja ima ekstrem za $x=3$ $\quad f(3) = 27 e^{-\frac{9}{6}} = 27 e^{-\frac{3}{2}} \approx 6$

$(-3,-6)$ i $(3,6)$ je maksimum f-je

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(x^2 e^{-\frac{x^2}{6}}\frac{1}{3}(9-x^2)\right)' = 2x e^{-\frac{x^2}{6}}\cdot\frac{1}{3}(9-x^2) + x^2 e^{-\frac{x^2}{6}}\cdot\left(-\frac{1}{6}\right)2x\cdot\frac{1}{3}(9-x^2) + x^2 e^{-\frac{x^2}{6}}\cdot\frac{1}{3}(-2x) =$$

$$= \frac{2}{3}x e^{-\frac{x^2}{6}}(9-x^2) - \frac{1}{9}x^3 e^{-\frac{x^2}{6}}(9-x^2) - \frac{2}{3}x^3 e^{-\frac{x^2}{6}} = x e^{-\frac{x^2}{6}}\left(\frac{2}{3}(9-x^2) - \frac{1}{9}x^2(9-x^2) - \frac{2}{3}x^2\right) = x e^{-\frac{x^2}{6}}\cdot\frac{54-6x^2-9x^2+x^4-6x^2}{9} = x e^{-\frac{x^2}{6}}\cdot\frac{x^4-21x^2+54}{9}$$

$y'' = 0$ akko $x = 0$ i $x^4 - 21x^2 + 54 = 0$

$x^2 = t$

$t^2 - 21t + 54 = 0$

$D = 441 - 216 = 225$

$t_{1,2} = \frac{21 \pm 15}{2}$

$t_1 = \frac{36}{2} = 18 \quad t_2 = \frac{6}{2} = 3$

$x^2 = 18 \qquad x^2 = 3$

$x = \pm\sqrt{18} \qquad x_3 = -\sqrt{3}$

$x_1 = 3\sqrt{2} \quad x_2 = -3\sqrt{2} \qquad x_4 = \sqrt{3} \approx 1{,}73$

$3\sqrt{2} \approx 4{,}24$

f-ja simetrična u odnosu na koordinatni početak pa nas zanima, u samo pozitivne vrijednosti

0 $\quad\sqrt{2}\quad$ $3\sqrt{2}$ ← prekidi y + nule y''

x	$(0, \sqrt{3})$	$(\sqrt{3}, 3\sqrt{2})$	$(3\sqrt{2}, +\infty)$
y''	$+$	$-$	$+$
y	∪	∩	∪

P.T. $\quad$ P.T. $\quad$ P.T.

$y = x^3 e^{-\frac{x^2}{6}}$

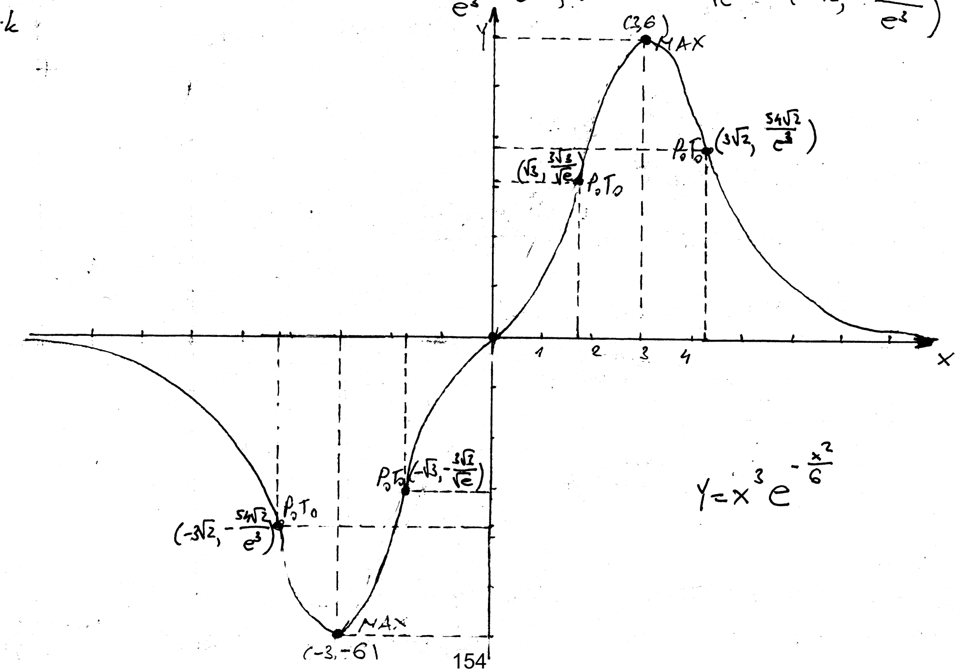
$y(0) = 0$

$y(\sqrt{3}) = 3\sqrt{3}\, e^{-\frac{3}{6}} = \frac{3\sqrt{3}}{\sqrt{e}} \approx 3{,}15$

$y(3\sqrt{2}) = 27\cdot 2\sqrt{2}\, e^{-\frac{9\cdot 2}{6}} = 54\sqrt{2}\, e^{-3} = \frac{54\sqrt{2}}{e^3} \approx 3{,}8$

Prevojne tačke su $(0,0)$, $\left(\sqrt{3}, \frac{3\sqrt{3}}{\sqrt{e}}\right)$, $\left(3\sqrt{2}, \frac{54\sqrt{2}}{e^3}\right)$, $\left(-\sqrt{3}, -\frac{3\sqrt{3}}{\sqrt{e}}\right)$ i $\left(-3\sqrt{2}, -\frac{54\sqrt{2}}{e^3}\right)$

grafik



(#) Ispitati i grafički predstaviti f-ju $y=\frac{1}{x}\ln x$.

Rj. definiciono područje

$x\neq 0,\quad x>0$

$\mathcal{D}: x\in(0,+\infty)$

parnost neparnost periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$

$\frac{1}{x}\ln x=0$

$\ln x=0$

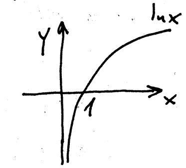
$x=e^0$

$x=1$

$(1,0)$ je nula f-je

$f(0)$ nije definisano

f-ja ne siječe y-osu



x	$(0,1)$	$(1,+\infty)$
$\ln x$	$-$	$+$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$$\lim_{x\to+0} f(x)=\lim_{x\to+0}\frac{1}{x}\ln x\ \left(=\infty\cdot(-\infty)\right)=\frac{1}{+0}\ln(+0)=(+\infty)\cdot(-\infty)=-\infty$$

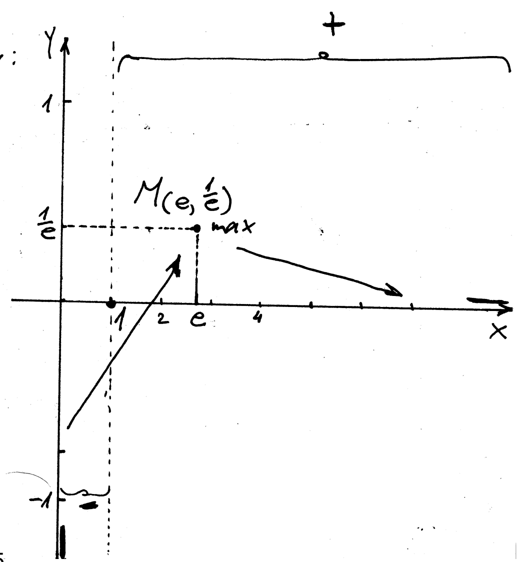
$\Rightarrow\quad x=0$ je V.A. (sa desne strane)

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\ln x}{x}\ \left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0\quad\Rightarrow$$

$\Rightarrow\quad y=0$ je H.A.

f-ja nema kosu asimptotu

počinjemo sa skiciranjem grafa:



rast i opadanje

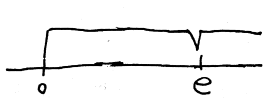
$$y'=\left(\frac{1}{x}\ln x\right)'=\left(\frac{\ln x}{x}\right)'=$$

$$=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

$y'=0$ akko $1-\ln x=0$

$\ln x=1$

$x=e\approx 2{,}7183$



nule y' + prekidi y

x	$(0,e)$	$(e,+\infty)$
y'	$+$	$-$
y	$\nearrow$	$\searrow$
	max	

rast i opadanje

$f(e)=\frac{1}{e}\ln e=\frac{1}{e}\approx 0{,}3679$

ekstremi f-je

Na osnovu tabele rasta i opadanja, f-ja ima maksimum u tački $M(e,\frac{1}{e})$.

prevojne tačke i intervali konveksnosti i konkavnosti.

$$y''=\left(\frac{1-\ln x}{x^2}\right)'=\frac{-\frac{1}{x}\cdot x^2-(1-\ln x)2x}{x^4}=\frac{-\cancel{x}-(1-\ln x)\cdot 2\cancel{x}}{x^{\cancel{4}\,3}}=\frac{-1-2+2\ln x}{x^3}$$

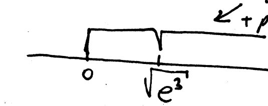
$y''=\frac{2\ln x-3}{x^3}$ $\qquad y''=0$ akko $2\ln x-3=0$

$2\ln x=3$

$\ln x=\frac{3}{2}$

$x=e^{\frac{3}{2}}=\sqrt{e^3}\approx 4{,}4817$

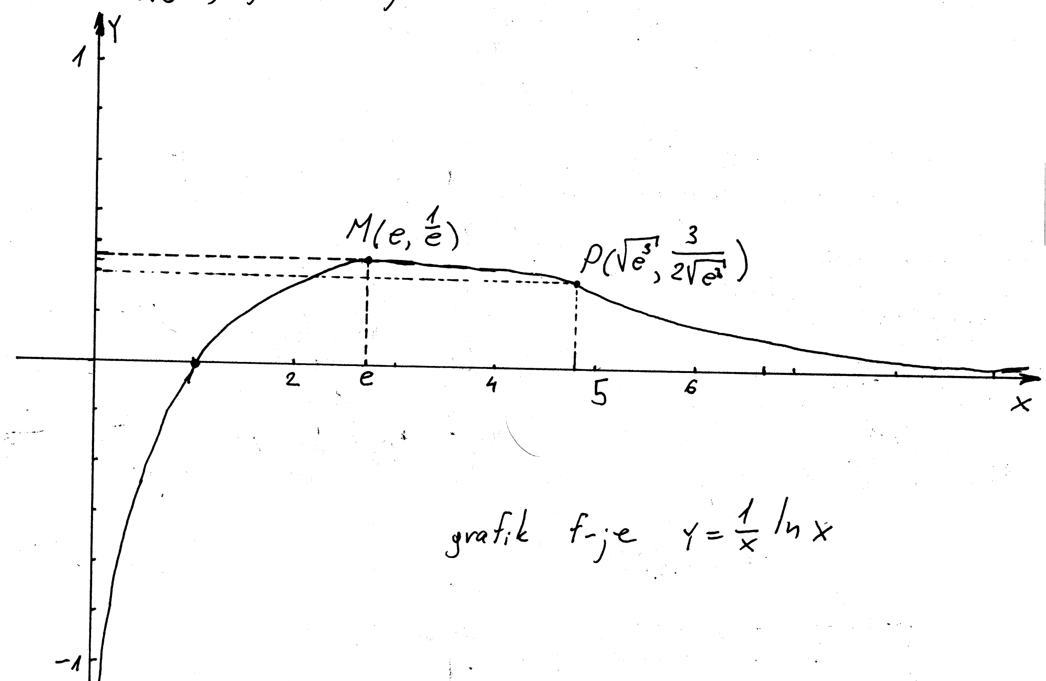
nule y'' + prekidi y



x	$(0,\sqrt{e^3})$	$(\sqrt{e^3},+\infty)$
y''	$-$	$+$
y	$\frown$	$\cup$
	P.T.	

$$f(e^{\frac{3}{2}})=\frac{1}{\sqrt{e^3}}\cdot\frac{3}{2}=\frac{3}{2\sqrt{e^3}}\approx 0{,}3347.$$

$P(\sqrt{e^3},\frac{3}{2\sqrt{e^3}})$ je prevojna tačka



grafik f-je $y=\frac{1}{x}\ln x$

Ispitati f-ju i nacrtati joj grafik $y=\frac{\ln x-1}{x^3}$.

Rj. definiciono područje

$x\neq 0 \quad x>0$

$\mathcal{D}: x\in(0,+\infty)$

parnost, neparnost, periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $\ln x-1=0$

$\ln x=1$

$x=e$

$(e,0)$ nula f-je

$e\approx 2{,}7183$

$f(0)=?$

$f(0)$ nije definisano

f-ja ne siječe y-osu

x	$(0,e)$	$(e,+\infty)$	
$\ln x-1$	$-$	$+$	
x^3	$+$	$+$	
y	$-$	$+$	znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$$\lim_{x\to+0} f(x)=\lim_{x\to+0}\frac{\ln x-1}{x^3}\left(=\frac{-\infty-1}{+0}\right)=\frac{-\infty}{+0}=-\infty \Rightarrow x=0 \text{ je V.A.}$$

(sa desne strane)

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{\ln x-1}{x^3}\left(=\frac{\infty}{\infty}\right)\stackrel{L.P.}{=}\lim_{x\to+\infty}\frac{\frac{1}{x}}{3x^2}=\lim_{x\to+\infty}\frac{1}{3x^3}=\frac{1}{\infty}=0$$

$\Rightarrow y=0$ je H.A.

f-ja nema K.A.

počinjemo sa skiciranjem grafa

rast i opadanje

$$y'=\left(\frac{\ln x-1}{x^3}\right)'=\frac{\frac{1}{x}\cdot x^{3}-(\ln x-1)\cdot 3x^{2}}{x^{6}}$$

$$y'=\frac{1-3\ln x+3}{x^4}=\frac{4-3\ln x}{x^4}$$

$y'=0$ akko $4-3\ln x=0$

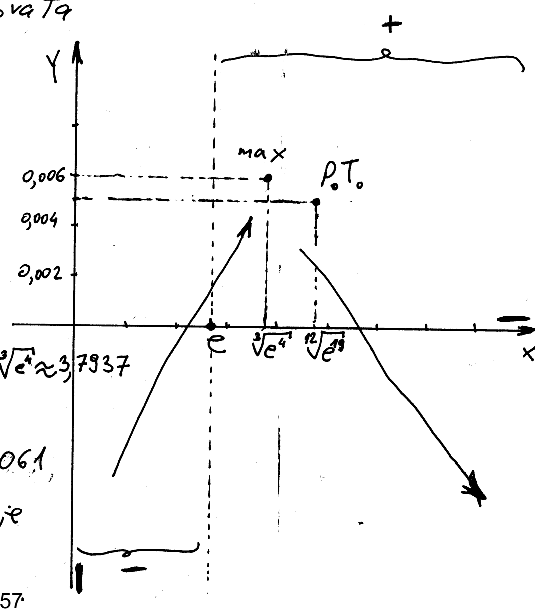
$3\ln x=4$

$\ln x=\frac{4}{3}$ $\quad x=e^{\frac{4}{3}}=\sqrt[3]{e^4}\approx 3{,}7937$

x	$(0,\sqrt[3]{e^4})$	$(\sqrt[3]{e^4},+\infty)$	
y'	$+$	$-$	
y	$\nearrow$	$\searrow$	rast i opadanje
	max		

$\frac{1}{3e^4}\approx 0{,}0061$

$$f(e^{\frac{4}{3}})=\frac{\ln e^{\frac{4}{3}}-1}{(\sqrt[3]{e^4})^3}=\frac{\frac{4}{3}-1}{(\sqrt[3]{e^4})^3}=\frac{\frac{1}{3}}{e^4}=\frac{1}{3e^4}$$



ekstremi f-je

na osnovu tabele rasta i opadanja tačka $M(\sqrt[3]{e^4},\frac{1}{3e^4})$ je tačka maksimuma.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\frac{4-3\ln x}{x^4}\right)'=\frac{-3\cdot\frac{1}{x}\cdot x^4-(4-3\ln x)\cdot 4x^3}{(x^4)^2}=\frac{-3\cancel{x^3}-(4-3\ln x)\cdot 4\cancel{x^3}}{x^5\cdot\cancel{x^3}}=\frac{-3-16+12\ln x}{x^5}$$

$$y''=\frac{12\ln x-19}{x^5}$$

$y''=0$ akko $12\ln x-19=0$

$12\ln x=19$

$\ln x=\frac{19}{12}$

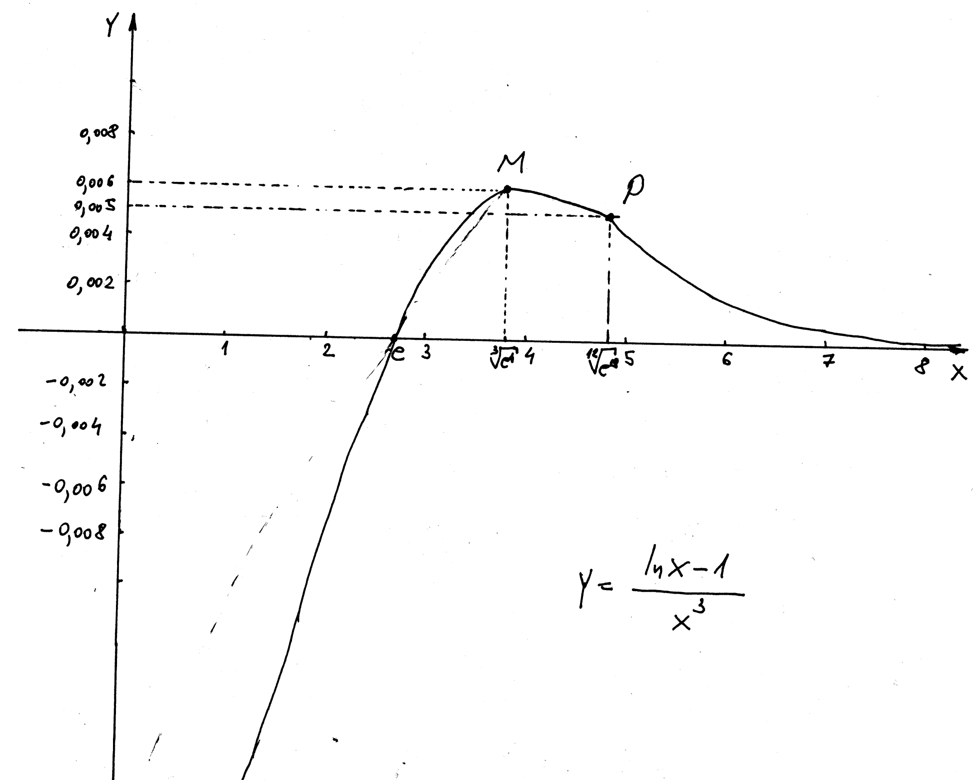
$x=e^{\frac{19}{12}}=\sqrt[12]{e^{19}}\approx 4{,}8712$

x	$(0,\sqrt[12]{e^{19}})$	$(\sqrt[12]{e^{19}},+\infty)$	
y''	$-$	$+$	intervali konveksnosti i konkavnosti
y	$\frown$	$\smile$	
	P.T.		

$$f(e^{\frac{19}{12}})=\frac{\ln e^{\frac{19}{12}}-1}{(e^{\frac{19}{12}})^3}=\frac{\frac{19}{12}-1}{e^{\frac{19}{4}}}=\frac{\frac{7}{12}}{e^{\frac{19}{4}}}=\frac{7}{12\sqrt[4]{e^{19}}}\approx 0{,}005$$

$P(\sqrt[12]{e^{19}},\frac{7}{12\sqrt[4]{e^{19}}})$ je prevojna tačka

grafik



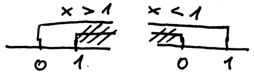
\# Ispitati f-ju i nacrtati joj grafik (bez analize drugog izvoda). $y=\frac{x}{x-1}\ln\frac{x}{x-1}$

Rj: definiciono područje

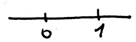
$x-1\neq 0$
$x\neq 1$

$\frac{x}{x-1}>0$

$x>0$ $x<0$
$x>1$ $x<1$



$\mathcal{D}: x\in(-\infty,0)\cup(1,+\infty)$



parnost, neparnost, periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $x=0$

za $x=0$ f-ja nije definisana

f-ja nema nulu i ne sječe y-osu

$\ln\frac{x}{x-1}>0$

$\ln\frac{x}{x-1}>\ln 1$

$\frac{x}{x-1}>1$

$\frac{x}{x-1}-1>0$

$\frac{x-x+1}{x-1}>0$

$\frac{1}{x-1}>0$

$x-1>0$

$x>1$

ponašanje na krajevima intervala definisanosti i asimptote

x	$(-\infty,0)$	$(1,+\infty)$
x	$-$	$+$
$x-1$	$-$	$+$
$\ln\frac{x}{x-1}$	$-$	$+$
y	$-$	$+$

znak f-je

$$\lim_{x\to 0-}f(x)=\lim_{x\to 0-}\frac{x}{x-1}\ln\frac{x}{x-1}\left(=(0-)\cdot(-\infty)\right)$$

$$=\lim_{x\to 0-}\frac{\ln\frac{x}{x-1}}{\frac{x-1}{x}}\left(=\frac{-\infty}{\infty}\right)\overset{LoPo}{=}\lim_{x\to 0-}\frac{\frac{1}{\frac{x}{x-1}}\left(\frac{x}{x-1}\right)'}{\left(\frac{x-x+1}{x^2}\right)}=\lim_{x\to 0-}\frac{\frac{\cancel{x-1}}{x}\cdot\frac{\cancel{x-1}-\cancel{x}}{(x-1)^{\cancel{2}}}}{\frac{1}{x^2}}=\lim_{x\to 0-}\frac{(-1)x^{\cancel{2}}}{\cancel{x}(x-1)}=0$$

nema V.A. za $x=0$

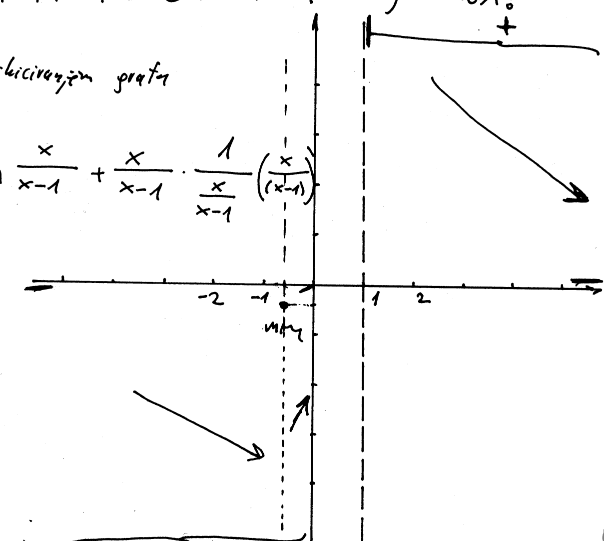
$$\lim_{x\to 1+0}f(x)=\lim_{x\to 1+0}\frac{x}{x-1}\ln\frac{x}{x-1}=(+\infty)\cdot(+\infty)=+\infty\Rightarrow x=1 \text{ je V.A.}$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{x}{x-1}\ln\frac{x}{x-1}=\lim_{x\to+\infty}\frac{1}{1-\frac{1}{x}}\ln\frac{1}{1-\frac{1}{x}}=1\cdot\ln 1=1\cdot 0=0\Rightarrow y=0 \text{ je H.A.}$$

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{1-\frac{1}{x}}\ln\frac{1}{1-\frac{1}{x}}=1\cdot\ln 1=1\cdot 0=0\Rightarrow y=0 \text{ je H.A.}$$

f-ja nema kose asimptote

nakon ovog koraka počinjemo sa skiciranjem grafa



rast i opadanje

$$y'=\left(\frac{x}{x-1}\ln\frac{x}{x-1}\right)'=\frac{x-1-x}{(x-1)^2}\ln\frac{x}{x-1}+\frac{x}{x-1}\cdot\frac{1}{\frac{x}{x-1}}\left(\frac{x}{x-1}\right)'$$

$$y'=\frac{-1}{(x-1)^2}\ln\frac{x}{x-1}+\frac{-1}{(x-1)^2}$$

$$y'=\frac{-1}{(x-1)^2}\left(\ln\frac{x}{x-1}+1\right)$$

$y'=0$ akko $\ln\frac{x}{x-1}+1=0$

$\ln\frac{x}{x-1}=-1$

$\frac{x}{x-1}=e^{-1}$

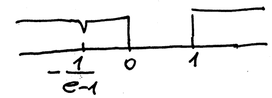
$\frac{x}{x-1}-\frac{1}{e}=0$

$\frac{ex-(x-1)}{e(x-1)}=0$

$ex-x+1=0$

$x(e-1)=-1$

$x=-\frac{1}{e-1}\approx-0{,}5820$



$\leftarrow$ prekidi y + nule y'

x	$(-\infty,-\frac{1}{e-1})$	$(-\frac{1}{e-1},0)$	$(1,+\infty)$
y'	$-$	$+$	$-$
y	$\searrow$	$\nearrow$	$\searrow$

rast i opadanje

$e>e^{-1}$

$e-1>e^{-1}-1$

$\frac{1}{e-1}<\frac{1}{e^{-1}-1}$ $\quad /\cdot(-1)$

$-\frac{1}{e-1}>-\frac{1}{e^{-1}-1}$

$\ln\frac{-\frac{1}{e^{-1}-1}}{-\frac{1}{e^{-1}-1}-1}=e$ $\quad \ln\frac{5}{4}\approx 0{,}22$

$\ln\frac{-\frac{1}{e-1}}{-\frac{1}{e-1}-1}=e^{-1}$

$$f\left(-\frac{1}{e-1}\right)=\frac{-\frac{1}{e-1}}{-\frac{1}{e-1}-1}\ln\frac{\frac{-1}{e-1}}{\frac{-1-(e-1)}{e-1}}=\frac{\frac{-1}{e-1}}{\frac{-e}{e-1}}\ln\frac{1}{e}=\frac{1}{e}\cdot(-1)=-\frac{1}{e}\approx-0{,}3679$$

ekstremi f-je

Na osnovu tabele rasta i opadanja tačka minimuma je $\left(-\frac{1}{e-1},-\frac{1}{e}\right)$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left[-(x-1)^{-2}\left(\ln\frac{x}{x-1}+1\right)\right]'=2(x-1)^{-3}\left(\ln\frac{x}{x-1}+1\right)+\left(-(x-1)^{-2}\right)\frac{x-1}{x}\cdot\frac{-1}{(x-1)^2}$$

$$y''=2(x-1)^{-3}\left(\ln\frac{x}{x-1}+1\right)-(x-1)^{-1}\cdot\frac{-1}{x(x-1)^2}=\frac{1}{(x-1)^3}\left[2\left(\ln\frac{x}{x-1}+1\right)+\frac{1}{x}\right]$$

bez analize drugog izvoda (crtamo graf)

$2\left(\ln\frac{x}{x-1}+1\right)+\frac{1}{x}=g(x)$

$g(-2)\approx 0{,}6891$

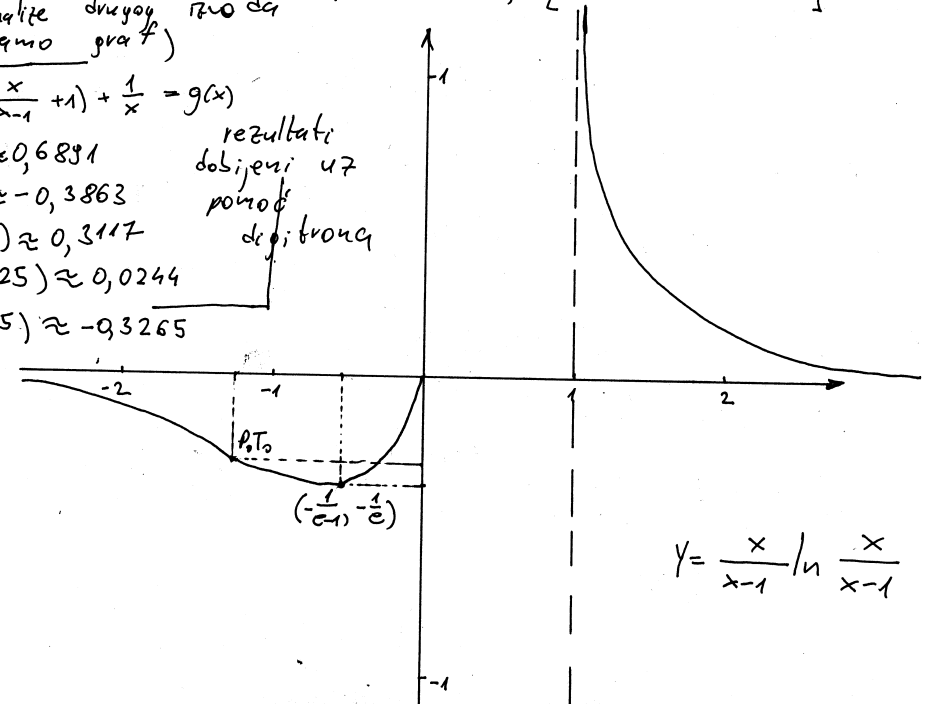
$g(-1)\approx -0{,}3863$

$g(-1{,}5)\approx 0{,}3117$

$g(-1{,}25)\approx 0{,}0244$

rezultati dobijeni uz pomoć digitrona

$f(-1{,}25)\approx -0{,}3265$



Ispitati f-ju i nacrtati njen grafik $Y=\frac{x^2+10}{x^2+4x+4}$.

R: $Y=\frac{x^2+10}{x^2+4x+4}=\frac{x^2+10}{(x+2)^2}$

definiciono područje

$x+2\neq 0$ $\quad \mathcal{D}: x\in(-\infty,-2)\cup(-2,+\infty)$

$x\neq -2$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom i znak f-je

$Y=0 \Rightarrow x^2+10=0$

Kako je $x^2+10>0\ \forall x\in\mathcal{D}$ to f-ja nema nule

$f(0)=\frac{0+10}{(0+2)^2}=\frac{10}{4}=\frac{5}{2}$

$(0,\frac{5}{2})$ je presjek sa Y-osom

$x^2+10>0\ \forall x\in\mathcal{D}$ f-ja je

$(x+2)^2>0\ \forall x\in\mathcal{D}$ uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$\lim\limits_{x\to-2-0} f(x)=\lim\limits_{x\to-2-0}\frac{x^2+10}{(x+2)^2}=\frac{(-2-0)^2+10}{(-2-0+2)^2}=\frac{14+0}{+0}=+\infty \Rightarrow x=-2$ je V. A. (sa lijeve strane)

$\lim\limits_{x\to-2+0} f(x)=\lim\limits_{x\to-2+0}\frac{x^2+10}{(x+2)^2}=\frac{(-2+0)^2+10}{(-2+0+2)^2}=\frac{14-0}{+0}=+\infty \Rightarrow x=-2$ je V. A. (sa desne strane)

$\lim\limits_{x\to\mp\infty} f(x)=\lim\limits_{x\to\mp\infty}\frac{x^2+10 \;:x^2}{x^2+4x+4 \;:x^2}=\lim\limits_{x\to\mp\infty}\frac{1+\frac{10}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}=1 \Rightarrow Y=1$ je H. A.

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

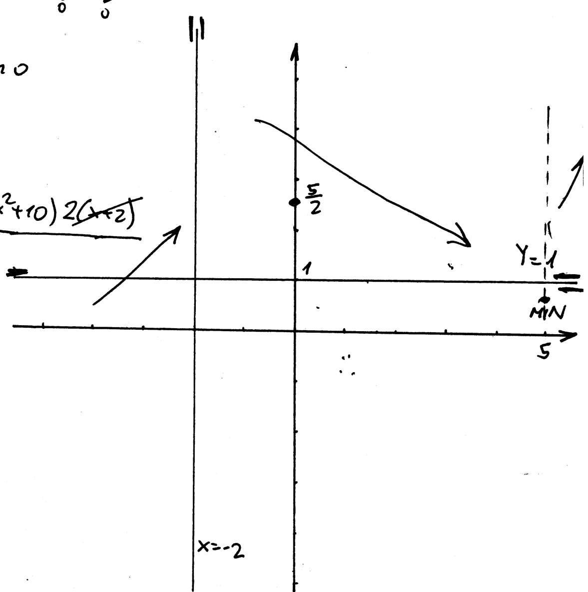
$Y'=\left(\frac{x^2+10}{(x+2)^2}\right)'=\frac{2x\cdot(x+2)^{\cancel{2}}-(x^2+10)2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}}$

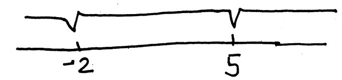
$Y'=\frac{\cancel{2x^2}+4x-\cancel{2x^2}-20}{(x+2)^3}$

$Y'=\frac{4x-20}{(x+2)^3}=4\frac{x-5}{(x+2)^3}$

$Y'=0$ akko $x-5=0$

$x=5$





x	$(-\infty,-2)$	$(-2,5)$	$(5,+\infty)$
Y'	+	−	+
Y	↗	↘	↗
			min

rast i opadanje

ekstremi f-je

Stacionarna tačka je $x=5$.

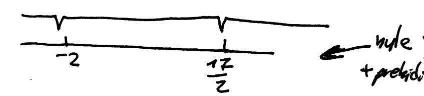
Na osnovu tabele rasta i opadanja vidimo da f-ja u toj tački ima ekstrem i to minimum

$f(5)=\frac{25+10}{7^2}=\frac{35}{49}\approx 0{,}71$ $\quad (5,\frac{35}{49})$ je tačka minimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$Y''=\left(4\frac{x-5}{(x+2)^3}\right)'=4\frac{1\cdot(x+2)^{\cancel{3}}-(x-5)3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}\,4}}=4\frac{x+2-3x+15}{(x+2)^4}$

$Y''=4\frac{-2x+17}{(x+2)^4}=-4\frac{2x-17}{(x+2)^4}$

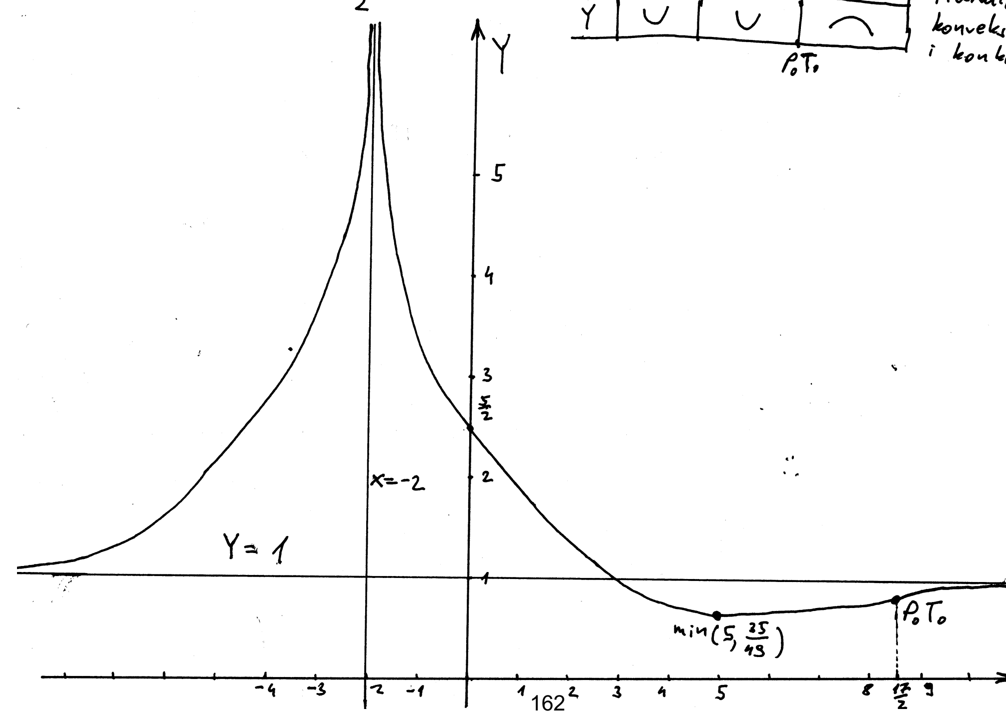


$Y''=0$ akko $2x-17=0$

$x=\frac{17}{2}$

x	$(-\infty,-2)$	$(-2,\frac{17}{2})$	$(\frac{17}{2},+\infty)$
Y''	+	+	−
Y	∪	∪	∩
		P. T.	

intervali konveksn. i konkavn.



Ispitati f-ju i nacrtati njen grafik: $y=\frac{x^3-2}{2x^2}$.

Rj. definiciono područje

$\mathcal{D}: x\neq 0$

parnost (neparnost), periodičnost

$$f(-x)=\frac{(-x)^3-2}{2(-x)^2}=\frac{-x^3-2}{2x^2}\neq \pm f(x)$$

· f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, ~~znak~~

$y=0$ akko $x^3-2=0$

$x=\sqrt[3]{2}\approx 1,26$

$(\sqrt[3]{2},0)$ je nula f-je

$f(0)$ = nije definisano

f-ja ne siječe y-osu

$2x^2>0 \quad \forall x\in\mathcal{D}$

$$\left.\begin{array}{ll} y>0 \text{ za } & x>\sqrt[3]{2} \\ y<0 \text{ za } & x<\sqrt[3]{2} \end{array}\right\} \text{znak f-je.}$$

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$$\left.\begin{array}{l} \lim\limits_{x\to 0-} f(x)=\lim\limits_{x\to 0-}\frac{x^3-2}{2x^2}=\frac{(0-)^3-2}{2(0-)^2}=\frac{-2-0}{0+}=-\infty \\ \lim\limits_{x\to 0+} f(x)=\lim\limits_{x\to 0+}\frac{x^3-2}{2x^2}=\frac{(0+)^3-2}{2(0+)^2}=\frac{-2+0}{+0}=-\infty \end{array}\right\} \Rightarrow x=0 \text{ je V. A.}$$

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^3-2}{2x^2}\,\frac{/:x^2}{/:x^2}=\pm\infty \qquad \text{f-ja nema H. A.}$$

Tražimo kosu asimptotu u obliku $y=kx+n$.

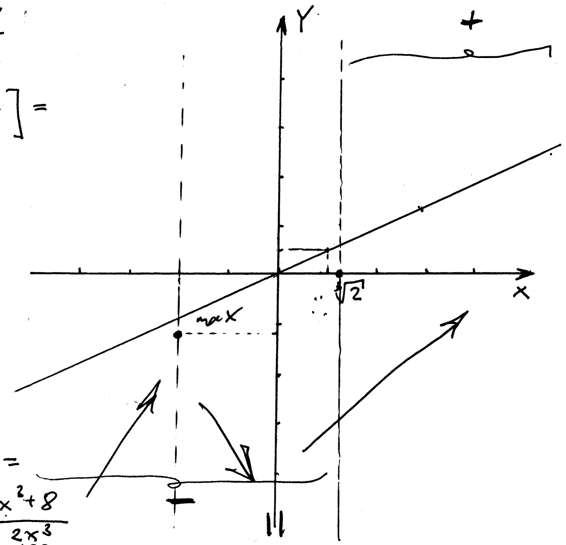
$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^3-2}{2x^3}\,\frac{/:x^3}{/:x^3}=\frac{1}{2}$$

$$n=\lim_{x\to\infty}[f(x)-kx]=\lim_{x\to\infty}\left[\frac{x^3-2}{2x^2}-\frac{1}{2}x\right]=$$

$$=\lim_{x\to\infty}\frac{x^3-2-x^3}{2x^2}=\lim_{x\to\infty}\frac{-2}{2x^2}=0$$

kosa asimptota je $y=\frac{1}{2}x$

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$$y'=\left(\frac{x^3-2}{2x^2}\right)'=\frac{3x^2\cdot 2x^2-(x^3-2)4x}{4x^4}=$$

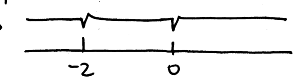
$$=\frac{6x^4-4x^4+8x}{4x^4}=\frac{2x^4+8x}{4x^4}=\frac{x^3+8}{2x^3}$$

$y'=\frac{x^3+8}{2x^3}$, $y'=0$ akko $x^3+8=0$

$x^3=-8$

$x=-2$

prekidi y + nule y'



x	$(-\infty,-2)$	$(-2,0)$	$(0,+\infty)$
y'	+	−	+
y	↗	↘	↗

· max N.D.

$$f(-2)=\frac{(-2)^3-2}{2(-2)^2}=\frac{-10}{8}=-\frac{5}{4}\approx -1,25$$

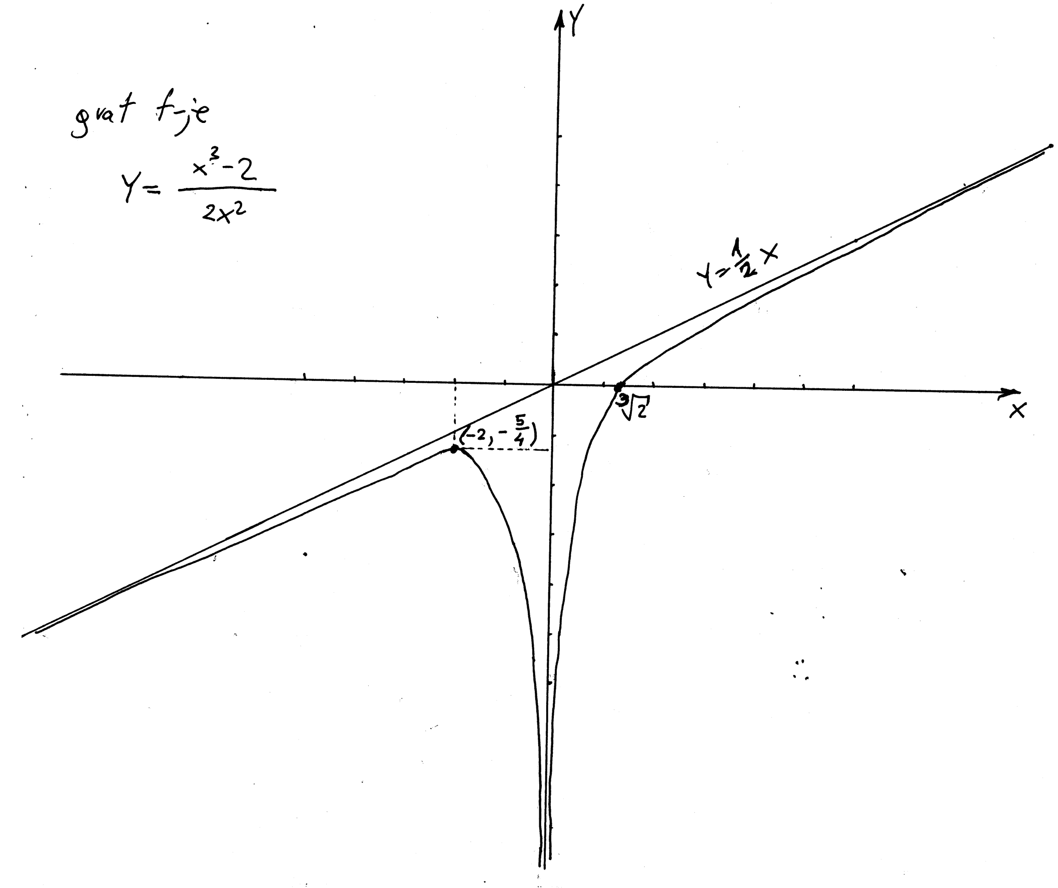
prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\frac{x^3+8}{2x^3}\right)'=\frac{3x^2\cdot 2x^3-(x^3+8)\cdot 6x^2}{4x^6}=\frac{6x^5-6x^5-48}{4x^6}=\frac{-48}{4x^6}=-\frac{12}{x^6}<0$$

F-ja nema prevojnih tački i uvijek je negativna što znači uvijek je $\cap$ oblika.

graf f-je

$y=\frac{x^3-2}{2x^2}$



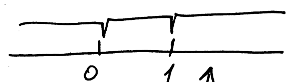
Ispitati f-ju i nacrtati njen grafik $y = e^{\frac{x}{1-x}} - 1$.

Rj. definiciono područje

$1-x \neq 0$
$x \neq 1$ $\quad \mathcal{D}: x \in (-\infty, 1) \cup (1, +\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$
f-ja nije ni parna ni neparna
f-ja nije periodična



nule od y + prekidi od y

nule, presjek sa y-osom, znak f-je

$y=0$ ako $e^{\frac{x}{1-x}} = 1$

tj. $\frac{x}{1-x} = 0 \Rightarrow x = 0$

(0,0) je nula f-je i presjek sa y-osom

$y > 0 \Leftrightarrow e^{\frac{x}{1-x}} - 1 > 0$

$e^{\frac{x}{1-x}} > 1$

$e^{\frac{x}{1-x}} > e^0$

$\frac{x}{1-x} > 0$

	$(-\infty, 0)$	$(0,1)$	$(1,+\infty)$
x	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
y	$-$	$+$	$-$

znak f-je

Ponašanje na krajevima intervala definisanosti i asimptote

za $x=1$ f-ja ima prekid

$$\lim_{x\to 1-0} f(x) = \lim_{x \to 1-0}\left(e^{\frac{x}{1-x}} - 1\right) = e^{\frac{1-0}{1-1+0}} - 1 = e^{\frac{1-0}{+0}} - 1 = e^{\infty} - 1 = \infty$$

$$\lim_{x\to 1+0} f(x) = \lim_{x \to 1+0}\left(e^{\frac{x}{1-x}} - 1\right) = e^{\frac{1+0}{1-1-0}} - 1 = e^{\frac{1+0}{-0}} - 1 = e^{-\infty} - 1 = \frac{1}{e^{\infty}} - 1 = -1$$

$x=1$ je vertikalna asimptota (sa lijeve strane)

$$\lim_{x\to\mp\infty} f(x) = \lim_{x\to\mp\infty}\left(e^{\frac{x}{1-x}} - 1\right) =$$

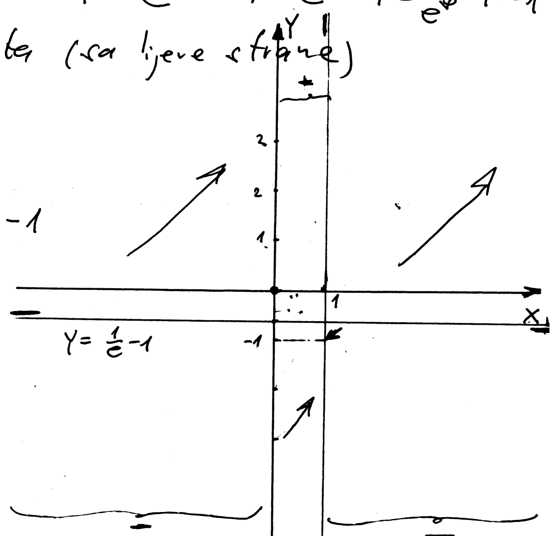
$$= \lim_{x\to\mp\infty}\left(e^{\frac{1}{\frac{1}{x} - 1}} - 1\right) = e^{-1} - 1 = \frac{1}{e} - 1$$

$y = \frac{1}{e} - 1 \approx -0{,}63$

je H. A.

kose asimptote nema

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je



rast i opadanje

$$y' = \left(e^{\frac{x}{1-x}} - 1\right)' = e^{\frac{x}{1-x}} \cdot \left(\frac{x}{1-x}\right)' = \frac{1\cdot(1-x) - x\cdot(-1)}{(1-x)^2} e^{\frac{x}{1-x}} = \frac{e^{\frac{x}{1-x}}}{(1-x)^2}$$

$y' = \frac{1}{(1-x)^2} e^{\frac{x}{1-x}}$ $\quad y' > 0$ za $\forall x \in \mathcal{D}$, f-ja $\nearrow$ za $\forall x$

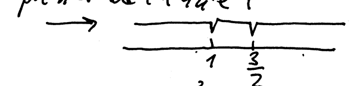
ekstremi f-je

$y' \neq 0 \ \forall x$ f-ja nema ekstrema

$$y'' = \left(\frac{1}{(1-x)^2} e^{\frac{x}{1-x}}\right)' = (-2)(1-x)^{-3}(-1) e^{\frac{x}{1-x}} + \frac{1}{(1-x)^2} \cdot \frac{1}{(1-x)^2} e^{\frac{x}{1-x}}$$

$y'' = \frac{2\cdot(1-x) + 1}{(1-x)^4} e^{\frac{x}{1-x}} = \frac{-2x+3}{(1-x)^4} e^{\frac{x}{1-x}}$ $\quad y'' = 0$ akko $x = \frac{3}{2}$

prekidi od y + nule y''



x	$(-\infty, 1)$	$(1, \frac{3}{2})$	$(\frac{3}{2}, +\infty)$
y''	$+$	$+$	$-$
y	$\cup$	$\cup$	$\cap$

P. T.

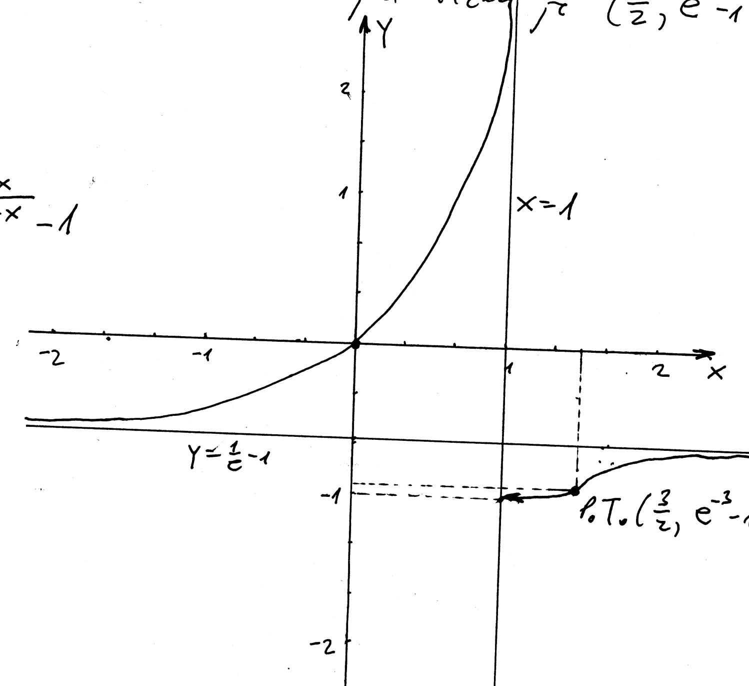
konveksnost i konkavnost

$f(\frac{3}{2}) = e^{\frac{\frac{3}{2}}{1-\frac{3}{2}}} - 1 = e^{\frac{\frac{3}{2}}{-\frac{1}{2}}} - 1$

$f(\frac{3}{2}) = e^{-3} - 1 \approx -0{,}95$

Prevojna tačka je $(\frac{3}{2}, e^{-3} - 1)$

graf f-je
$y = e^{\frac{x}{1-x}} - 1$



Ispitati f-ju i nacrtati njen grafik: $y=\frac{\ln^2 x+1}{x^2}$.

Rj.: definiciono područje

$x\neq 0$; $x>0$

$D:\ x\in(0,+\infty)$

parnost (neparnost), periodičnost

. D nije simetrično

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $\ln^2x+1=0$

$(\ln x)^2=-1$

f-ja nema nulu

$f(0)$ nije definisano

f-ja ne siječe y-osu

$\ln^2x+1>0\ \forall x\in D$

$x^2>0\ \forall x\in D$

f-ja je uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x\leq 0$ f-ja nije definisana

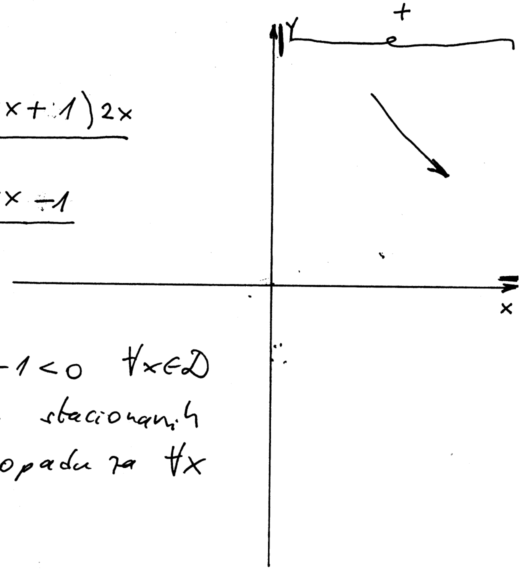
$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{\ln^2x+1}{x^2}=\frac{+\infty}{0+}=+\infty \Rightarrow x=0 \text{ je vertikalna asimptota}$$

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{\ln^2x+1}{x^2}\left(=\frac{\infty}{\infty}\right)\stackrel{L.P.}{=}\lim_{x\to+\infty}\frac{2\ln x\cdot\frac{1}{x}}{2x}=\lim_{x\to+\infty}\frac{\ln x}{x^2}\left(\frac{\infty}{\infty}\right)\stackrel{L.P.}{=}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=0 \Rightarrow y=0 \text{ je horizontalna asimptota}$$

f-ja nema kosu asimptotu

počinjemo skicirati grafik



rast i opadanje

$$y'=\left(\frac{\ln^2x+1}{x^2}\right)'=\frac{2\ln x\cdot\frac{1}{x}\cdot x^2-(\ln^2x+1)2x}{x^4}$$

$$=\frac{2x(\ln x-\ln^2x-1)}{x^4}=2\,\frac{\ln x-\ln^2x-1}{x^3}$$

$y'=0$ akko $-\ln^2x+\ln x-1=0$

$\ln x=t$

$-t^2+t-1=0$

$t^2-t+1=0$

$D=1-4<0$

$-\ln^2x+\ln x-1<0\ \forall x\in D$

f-ja nema stacionarnih tački i opada za $\forall x$

ekstremi f-je

f-ja nema stacionarnih tački $\Rightarrow$ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=2\left(\frac{\ln x-\ln^2x-1}{x^3}\right)'=2\,\frac{(\frac{1}{x}-2\ln x\cdot\frac{1}{x})\cancel{x^3}^{2}-(\ln x-\ln^2x-1)\cdot 3\cancel{x^2}}{\cancel{x^6}x^4}=$$

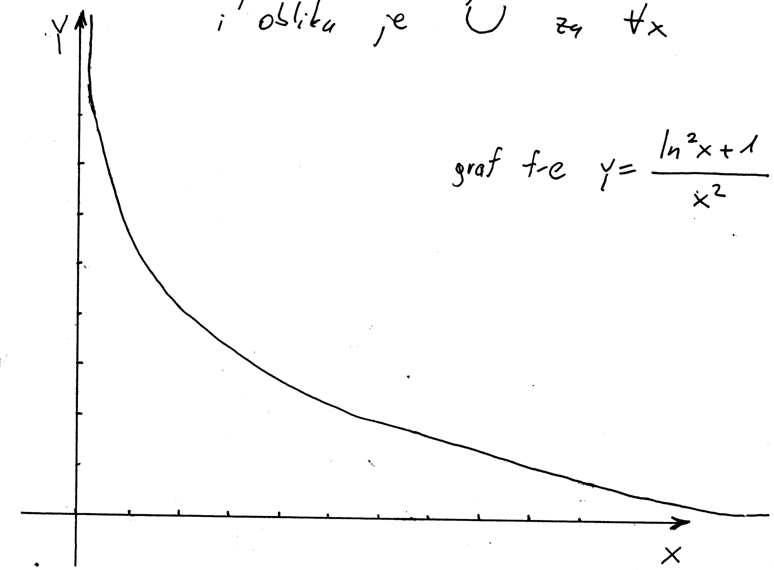
$$=2\,\frac{1-2\ln x-3\ln x+3\ln^2x+3}{x^4}=2\,\frac{3\ln^2x-5\ln x+4}{x^4}$$

$3\ln^2x-5\ln x+4=0$

$\ln x=t$ $\quad 3t^2-5t+4=0$ $\quad\Rightarrow\quad 3\ln^2x-5\ln x+4>0\ \forall x$

$D=25-48<0$ $\quad x^4>0\ \forall x$

$y''>0\ \forall x\in D \Rightarrow$ f-ja nema prevojnih tački i oblika je $\cup$ za $\forall x$



graf f-je $y=\frac{\ln^2x+1}{x^2}$

(#) Ispitati f-ju i nacrtati joj grafik $y=\frac{x^4-9x^2+12}{3x}$.

R: definiciono područje

$D: x\neq 0$

$x\in R\setminus\{0\}$

parnost (neparnost), periodičnost

$f(-x)=\frac{(-x)^4-9(-x)^2+12}{3(-x)} = -\frac{x^4-9x^2+12}{3x}=-f(x)$

f-ja je neparna $\xrightarrow{\text{simetrična u odnosu na } (0,0)}$ (dovoljno je je ispitati za $x>0$)

f-ja nije periodična

znak f-je

0, 1,2758, 3,71 ← prekidi f-je y + nule f-je y

x	(0, 1,27)	(1,27, 3,71)	(3,71, +∞)
y	+	−	+

znak f-je

nule, presjek sa y-osom i znak f-je

$y=0$ akko $x^4-9x^2+12=0$

$x^2=t$

$t^2-9t+12=0$

$D=81-48=33$

$t_{1,2}=\frac{9\pm\sqrt{33}}{2}$

$x^2=\frac{9-\sqrt{33}}{2}$ $x^2=\frac{9+\sqrt{33}}{2}$

$x_1\approx -1,2758$ $x_3\approx -2,7152$

$x_2\approx 1,2758$ $x_4\approx 2,7152$

$f(0)$ nije definisano

f-ja ne siječe y-osu

ponašanje na krajevima intervala definisanosti i asimptote

Za $x=0$ f-ja ima prekid

$\lim_{x\to 0+} f(x)=\lim_{x\to +0}\frac{x^4-9x^2+12}{3x}=\frac{12}{0+}=+\infty$

$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{x^4-9x^2+12}{3x}=\frac{12}{0-}=-\infty$

$\Rightarrow x=0$ je V. A.

$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^4-9x^2+12}{3x}\,/:x=\lim_{x\to\pm\infty}\frac{x^3-9x+\frac{12}{x}}{3}=\pm\infty \Rightarrow$ f-ja nema H. A.

tražimo kosu asimptotu u obliku $y=kx+n$,

$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^4-9x^2+12}{3x^2}\,/:x^2=\infty$

f-ja nema kosu asimptotu

Nakon ovog koraka počinjemo skicirati graf f-je.

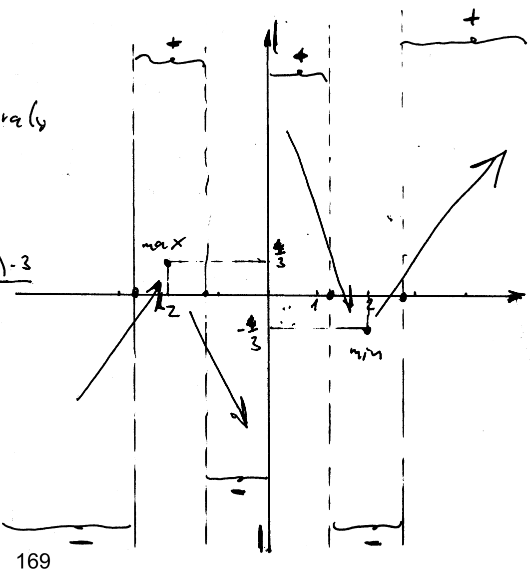
rast i opadanje

$y'=\left(\frac{x^4-9x^2+12}{3x}\right)'=\frac{(4x^3-18x)3x-(x^4-9x^2+12)\cdot 3}{9x^2}$

$=\frac{12x^4-54x^2-3x^4+27x^2+36}{9x^2}=$

$=\frac{9x^4-27x^2-36}{9x^2}=\frac{x^4-3x^2-4}{x^2}$

$y'=x^2-3-\frac{4}{x^2}$



$y'=0$ akko $x^4-3x^2-4=0$

$t=x^2$

$t^2-3t-4=0$

$D=9+16=25$

$t_{1,2}=\frac{3\pm 5}{2}$

$t_1=-1$ $t_2=4$

$\Downarrow$

$x^2=4$

$x_1=-2$ $x_2=2$

0, 2 ← prekidi f-je y + nule f-je y'

x	(0,2)	(2,+∞)
y'	−	+
y	↘	↗

min

$f(2)=\frac{16-36+12}{6}$

$f(2)=-\frac{8}{6}=-\frac{4}{3}$

ekstremi f-je

Na osnovu tabele rasta i opadanja i simetričnosti grafa f-ja ima minimum u $(2,-\frac{4}{3})$ i maksimum u $(-2,\frac{4}{3})$.

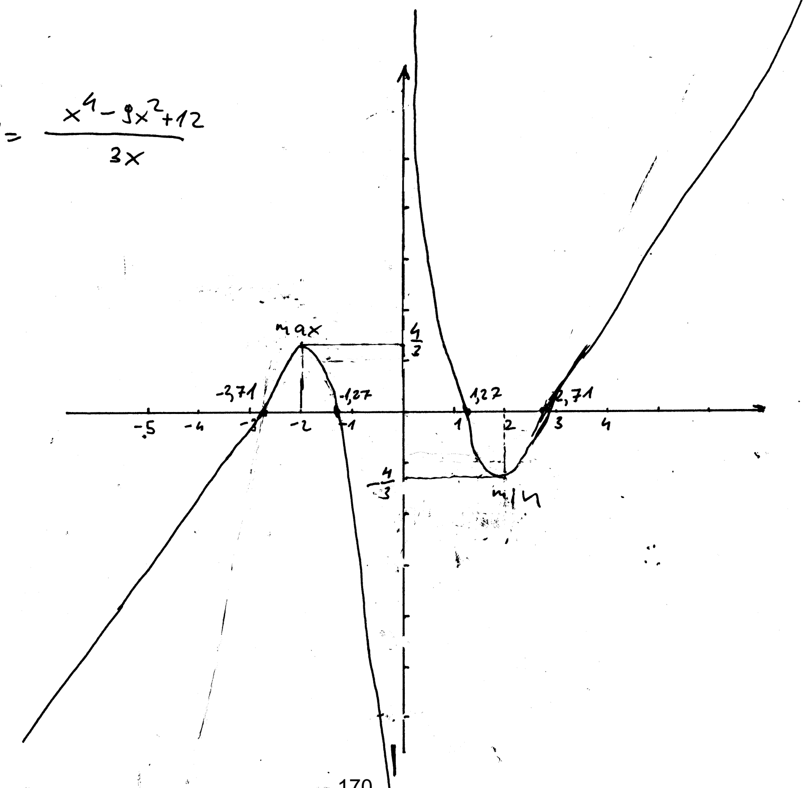
prevojne tačke i intervali konveksnosti i konkavnosti

$y''=\left(x^2-3-\frac{4}{x^2}\right)'=2x-4(-2)x^{-3}=2x+\frac{8}{x^3}$

$y''=\frac{2x^4+8}{x^3}$ kako je $2x^4+8>0\ \forall x \Rightarrow$ f-ja nema prevojnih tačaka

za $x<0$ ∩, a za $x>0$ ∪

f-ja $y=\frac{x^4-9x^2+12}{3x}$



(#) Ispitati f-ju $y=\frac{ax+b}{x^2+x+1}$ i nacrtati joj grafik ako se zna da ona ima ekstrem u tački $T(1,\frac{2}{3})$.

Rj. $f(x)=\frac{ax+b}{x^2+x+1}$

$f(1)=\frac{2}{3} \Rightarrow \frac{a+b}{3}=\frac{2}{3}$

$a+b=2$

$$y'=\frac{a(x^2+x+1)-(ax+b)(2x+1)}{(x^2+x+1)^2}$$

$$y'=\frac{a(x^2+x+1)-(2ax^2+ax+2bx+b)}{(x^2+x+1)^2}$$

$$y'=\frac{-ax^2-2bx+a-b}{(x^2+x+1)^2}$$

U stacionarnoj tački f-ja može imati ekstrem

$y'=0 \Rightarrow -ax^2-2bx+a-b=0$

$x=1$

$-a-2b+a-b=0$

$-3b=0$

$b=0, \quad a=2$

$$y=\frac{2x}{x^2+x+1}$$

$$y'=\frac{-2x^2+2}{(x^2+x+1)^2}$$

$$y'=(-2)\frac{x^2-1}{(x^2+x+1)^2}$$

definiciono područje

$x^2+x+1\neq 0$

f-ja je definisana za $\forall x$

nule, presjek sa x-osom, znak f-je

$y=0 \Rightarrow 2x=0 \Rightarrow x=0$

$(0,0)$ je presjek sa y-osom i nula f-je

kako je $x^2+x+1>0 \ \forall x$ to je

$y>0$ za $x>0$

$y<0$ za $x<0$ znak f-je

parnost (neparnost), periodičnost

$f(-x)=\frac{-2x}{x^2-x+1}$

f-ja nije ni parna ni neparna

f-ja nije periodična

ponašanje na krajevima intervala definisanosti i asimptote

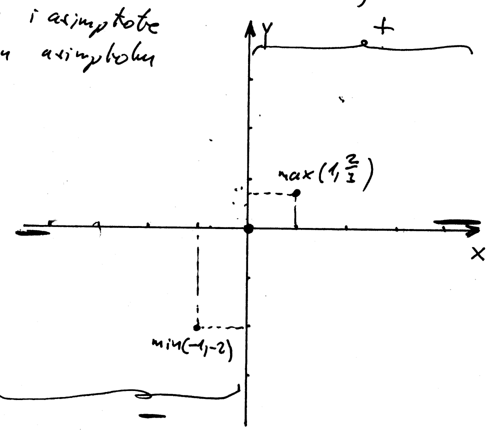
f-ja nema prekid $\Rightarrow$ f-ja nema vertikalnu asimptotu

$$\left.\begin{aligned}\lim_{x\to\infty} f(x)&=\lim_{x\to\infty}\frac{2x \ /:x}{x^2+x+1 \ /:x}=0\\ \lim_{x\to-\infty} f(x)&=\lim_{x\to-\infty}\frac{2x}{x^2+x+1}=0\end{aligned}\right\}\Rightarrow$$

$\Rightarrow x=0$ je H.A.

F-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik f-je.



rast i opadanje

$$y'=(-2)\frac{x^2-1}{(x^2+x+1)^2}$$

$y'=0 \Rightarrow x=\pm 1$

x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
y'	−	+	−
y	↘	↗	↘

min max

ekstremi f-je

$f(-1)=\frac{-2}{1-1+1}=-2$

$f(1)=\frac{2}{1+1+1}=\frac{2}{3}$

F-ja ima minimum u tački $P(-1,-2)$ i maksimum u tački $(1,\frac{2}{3})$

prevojne tačke i intervali konveksnosti i konkavnosti

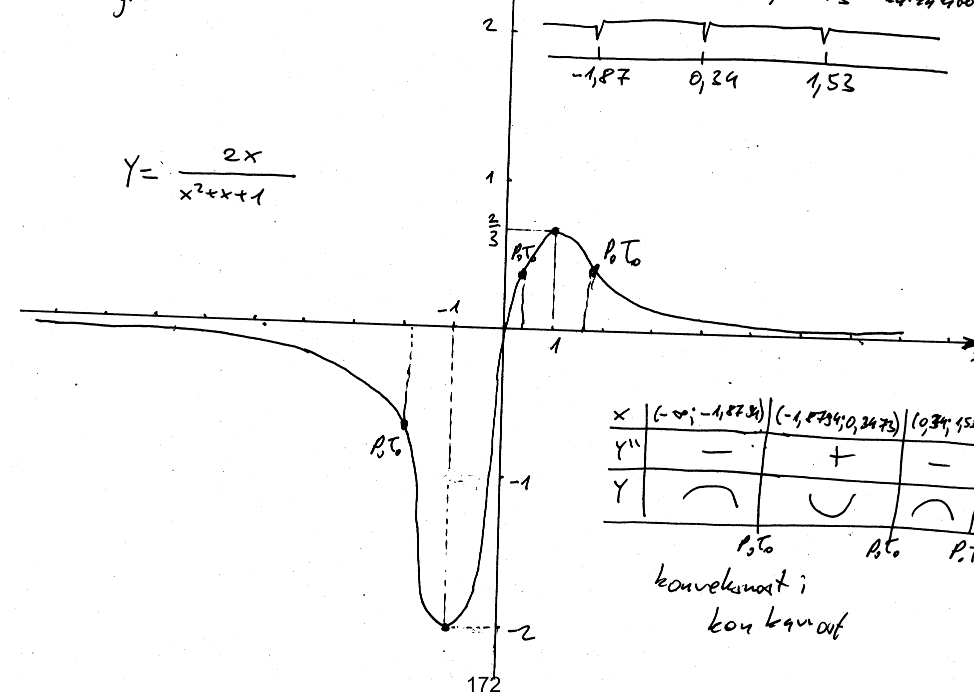
$$y''=(-2)\left(\frac{x^2-1}{(x^2+x+1)^2}\right)'=(-2)\frac{2x(x^2+x+1)^{\cancel{2}\,1}-(x^2-1)\,2\cancel{(x^2+x+1)}(2x+1)}{(x^2+x+1)^{\cancel{4}\,3}}$$

$$y''=(-2)\frac{2x^3+\cancel{2x^2}+2x-4x^3+4x-\cancel{2x^2}+2}{(x^2+x+1)^3}=(-2)\frac{-2x^3+x+2}{(x^2+x+1)^3}=(-2)\frac{(-2)(x^3-3x-1)}{(x^2+x+1)^3}$$

$$y''=4\frac{x^3-3x-1}{(x^2+x+1)^3}$$

korjeni od x^3-3x-1 su $x_1=-1{,}8794$, $x_2=1{,}5321$, $x_3=0{,}3473$ (vrijednosti izračunate uz pomoć kalkulatora)

crtamo grafik



x	$(-\infty;-1{,}8794)$	$(-1{,}8794;0{,}3473)$	$(0{,}34;1{,}53)$	[illegible]
y''	−	+	−	+
y	∩	∪	∩	∪

P.T. P.T. P.T.

konveksnost i konkavnost

(#) Ispitati f-ju i nacrtati joj grafik $y = x e^{\frac{1}{2}(1-\frac{1}{x^2})}$

Rj. definiciono područje

$x \neq 0$

$D: x \in R \setminus \{0\}$

parnost (neparnost), periodičnost

$f(-x) = -x e^{\frac{1}{2}(1-\frac{1}{(-x)^2})} = -x e^{\frac{1}{2}(1-\frac{1}{x^2})} = -f(-x)$

f-ja je neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$f(0)$ nije definisano

f-ja ne siječe y-osu

$y \neq 0, \ \forall x \in D$

$(e^{\frac{1}{2}(1-\frac{1}{x^2})} > 0 \ \forall x)$

f-ja nema nulu

x	$(-\infty, 0)$	$(0, +\infty)$
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (0-)\cdot e^{\frac{1}{2}(1-\infty)} = (0-)e^{-\infty} = \frac{0-}{e^{\infty}} = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (0+)e^{-\infty} = 0$$

f-ja nema vertikalnu asimptotu

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (-\infty)\cdot e^{\frac{1}{2}} = -\infty$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = \infty \cdot e^{\frac{1}{2}} = \infty$$

f-ja nema horizontalnu asimptotu

tražimo kosu asimptotu u obliku

$y = kx + n$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{2}(1-\overset{\to 0}{\frac{1}{x^2}})} = e^{\frac{1}{2}} = \sqrt{e}$$

$$n = \lim_{x\to\infty} [f(x) - k\cdot x] = \lim_{x\to\infty} (x e^{\frac{1}{2}(1-\frac{1}{x^2})} - e^{\frac{1}{2}}x)$$

$$= \lim_{x\to\infty} x(e^{\frac{1}{2}(1-\frac{1}{x^2})} - e^{\frac{1}{2}}) =$$

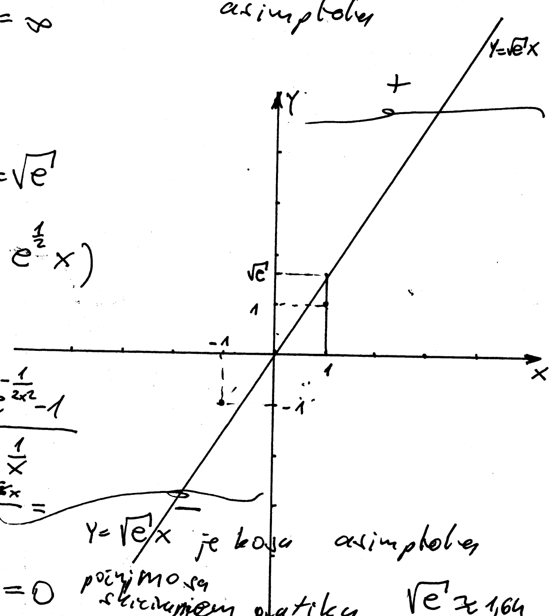
$$= \lim_{x\to\infty} e^{\frac{1}{2}} x (e^{\frac{-1}{2x^2}} - 1) = \sqrt{e} \lim_{x\to\infty} \frac{e^{-\frac{1}{2x^2}} - 1}{\frac{1}{x}}$$

$$\left(=\frac{0}{0}\right) \overset{L.P.}{=} \sqrt{e} \lim_{x\to\infty} \frac{e^{-\frac{1}{2x^2}} \cdot (-\frac{1}{2})(-2)\frac{1}{x^3}}{\frac{-1}{x^2}} =$$

$$= \sqrt{e} \lim_{x\to\infty} \frac{-e^{-\frac{1}{2x^2}}}{x} = \sqrt{e} \cdot \frac{-1}{\infty} = 0$$

$y = \sqrt{e}x$ je kosa asimptota

počinjemo sa skiciranjem grafika $\sqrt{e} \approx 1{,}64$



rast i opadanje

$$\left(-\frac{1}{x^2}\right)' = (-x^{-2})' = 2x^{-3} = \frac{2}{x^3}$$

$$y' = (x e^{\frac{1}{2}(1-\frac{1}{x^2})})' = e^{\frac{1}{2}(1-\frac{1}{x^2})} + x e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \left(\frac{1}{2}(1-\frac{1}{x^2})\right)' =$$

$$= e^{\frac{1}{2}(1-\frac{1}{x^2})} + x e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{x^3} = e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(1 + \frac{1}{x^2}\right)$$

$y' = 0$ akko $1 + \frac{1}{x^2} = 0$

$\frac{x^2+1}{x^2} = 0$

$y' > 0 \ \forall x \Rightarrow$ f-ja uvijek raste

f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left[e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(1+\frac{1}{x^2}\right)\right]' = e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{x^3}\left(1+\frac{1}{x^2}\right) + e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{-2}{x^3} =$$

$$= e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(\frac{1}{x^3} + \frac{1}{x^5} - \frac{2}{x^3}\right) = \left(\frac{1}{x^5} - \frac{1}{x^3}\right) e^{\frac{1}{2}(1-\frac{1}{x^2})}$$

$f(1) = 1 e^{\frac{1}{2}0} = 1$

$y'' = 0$ akko $\frac{1-x^2}{x^5} = 0 \Rightarrow 1 - x^2 = 0$

$x = \pm 1$

prekidi od f + nule od y''

$-1 \quad 0 \quad 1$

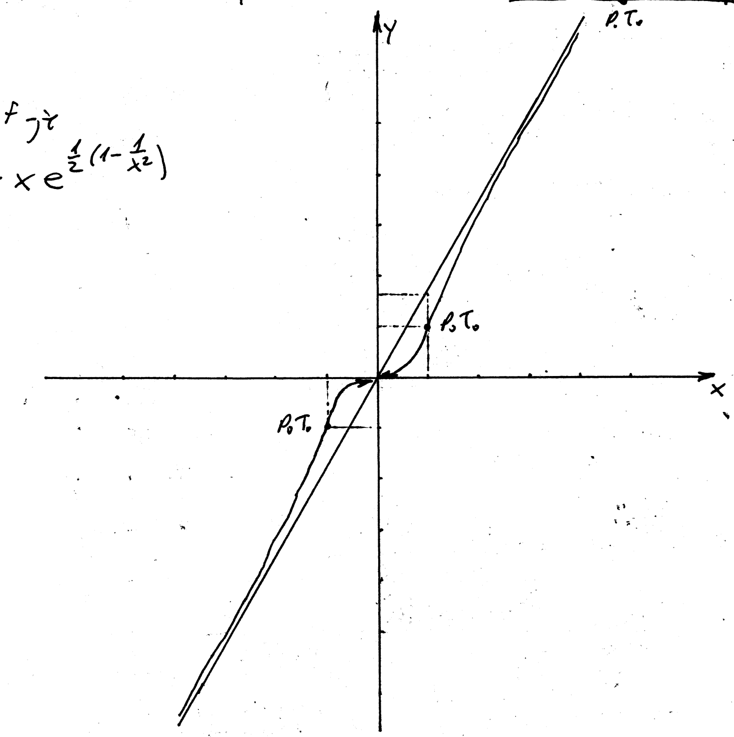
	$(0,1)$	$(1,+\infty)$
y''	+	−
y	∪	∩

P.T.

$(1,1)$ i $(-1,-1)$ su prevojne tačke

graf. f-je

$y = x e^{\frac{1}{2}(1-\frac{1}{x^2})}$



(#) Ispitati f-ju i nacrtati joj grafik $y=\ln\frac{x^2-3x+2}{x^2+1}$.

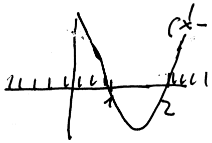
Rj. definiciono područje

Kako je $x^2+1>0 \quad \forall x\in\mathbb{R}$

to iz $\frac{x^2-3x+2}{x^2+1}>0 \Rightarrow$

treba da bude $x^2-3x+2>0$

$(x-1)(x-2)>0$



$\mathcal{D}: x\in(-\infty,1)\cup(2,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

ponašanje na krajevima intervala definisanosti i asimptote

f-ja ima prekid za $x=1$ i $x=2$

$$\lim_{x\to 1-0} f(x)=\lim_{x\to 1-0}\ln\frac{x^2-3x+2}{x^2+1}=\ln\frac{(1-0)^2-3(1-0)+2}{(1-0)^2+1}=\ln(0_+)=-\infty \Rightarrow$$

$\Rightarrow x=1$ je V. A. (sa lijeve str.)

$$\lim_{x\to 2+0} f(x)=\lim_{x\to 2+0}\ln\frac{x^2-3x+2}{x^2+1}=\ln(0_+)=-\infty$$

$\Rightarrow x=2$ je V. A. (sa desne strane)

$$\lim_{x\to\mp\infty} f(x)=\lim_{x\to\mp\infty}\ln\frac{x^2-3x+2}{x^2+1}=\lim_{x\to\mp\infty}\ln\frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{1}{x^2}}=\ln 1=0$$

$\Rightarrow y=0$ je H. A.

K. A. nema

počinjemo sa skiciranjem grafa

rast i opadanje

$$y'=\frac{x^2+1}{x^2-3x+2}\cdot\left(\frac{x^2-3x+2}{x^2+1}\right)'$$

$$y'=\frac{\cancel{x^2+1}}{x^2-3x+2}\cdot\frac{(2x-3)(x^2+1)-(x^2-3x+2)\cdot 2x}{(x^2+1)^{\cancel{2}}}=$$

$$=\frac{\cancel{2x^3}+2x-3x^2-3-\cancel{2x^3}+6x^2-4x}{(x^2-3x+2)(x^2+1)}=\frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}$$

nule, presjek sa y-osom, znak

$y=0 \Rightarrow \ln\frac{x^2-3x+2}{x^2+1}=0$

$\Rightarrow \frac{x^2-3x+2}{x^2+1}=1 \quad /\cdot x^2+1$

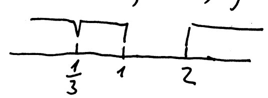
$\cancel{x^2}-3x+2=\cancel{x^2}+1$

$3x=1 \Rightarrow x=\frac{1}{3}$

$(\frac{1}{3},0)$ je nula f-je

$y(0)=\ln 2\approx 0,6931$

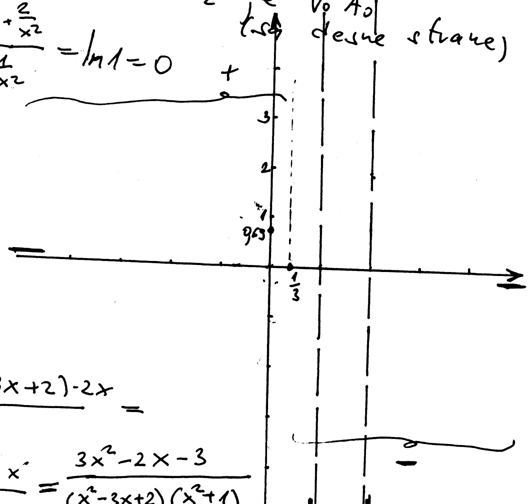
$(0,\ln 2)$ je presjek sa y-osom



$\leftarrow$ prekidi y + nule y

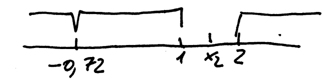
x	$(-\infty,\frac{1}{3})$	$(\frac{1}{3},1)$	$(2,+\infty)$
y	+	−	−

Znak f-je



$y'=0 \Rightarrow 3x^2-2x-3=0 \Rightarrow x_{1,2}=\frac{2\pm\sqrt{4+36}}{6}$

$x_{1,2}=\frac{2\pm 2\sqrt{10}}{6}=\frac{1\pm\sqrt{10}}{3}$



$x_1=\frac{1+\sqrt{10}}{3}\approx 1,387\notin\mathcal{D}$

$x_2=\frac{1-\sqrt{10}}{3}\approx -0,721\in\mathcal{D}$

x	$(-\infty,\frac{1-\sqrt{10}}{3})$	$(\frac{1-\sqrt{10}}{3},1)$	$(2,+\infty)$
y'	+	−	+
y	↗	↘	↗

max

ekstremi f-je

$f(\frac{1-\sqrt{10}}{3})\approx 1,016$

F-ja ima maksimum u tački $(-0,72;\ 1,02)$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}\right)'=\underset{\dots}{\text{ZA VJEŽBU}}=\frac{-6x^5+15x^4-30x^2+30x-13}{(x^2-3x+2)^2(x^2+1)^2}$$

$y''=0$ akko $x=-1,5166$ (izračunato uz pomoć kalkulatora)

Kako je brojnik u y'' previše složen nije potrebno praviti tabelu konveksnosti i konkavnosti

grafik f-je

$y=\ln\frac{x^2-3x+2}{x^2+1}$